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Shape Control of the Cubic Trigonometric B-Spline Polynomial Curve with A Shape Parameter

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Abstract

A cubic Trigonometric Polynomial B-spline curve with a shape parameter is presented in this paper. Each curve segment is generated by four consecutive control points. The shape of the curve can be adjusted by altering the values of shape parameters while the control polygon is kept unchanged. These curves are closer to the control polygon than the cubic Bézier curves, for all values of shape parameter. With the increase of the shape parameter, the curve approaches to the control polygon. Finally we study B-spline surfaces with a shape parameter.

Keywords: Trigonometric Bézier Basis Function, Trigonometric Bézier Curve, Shape Parameter, Cubic Trigonometric Polynomial B-Spline Curve, Continuity

Introduction

The importance of Trigonometric B-spline curves and surfaces in Computer Aided Geometric Designing (CAGD) and Computer Graphics (CG) is well known. Curves describing engineering objects are generally smooth and well behaved. Products such as car bodies, ship hulls, airplane fuse lane and wings, propeller blades, shoe insoles and bottles are a few examples that require free form curves and surfaces. Therefore to fulfill these requirements, parametric representation of curves and surfaces is widely used in the field of CAGD and CG. The type of input data and its influence on the control of the resulting curve determine the use and effectiveness of curve in design. Designers need a curve representation that is directly related to the control points and is flexible enough to bend, twist or change the curve shape by changing one or more control points. For these reasons, Bézier curve and surface plays a significant role in the field of CAGD and CG. However, they have many shortcomings due to polynomial forms. In particular, they cannot represent exactly transcendental curves such as the helix and the cycloid etc. To overcome the shortcomings, many bases are presented using trigonometric functions or the blending of polynomial and trigonometric functions in [1-6].

Some of these existing methods have no shape parameters; hence the shape of the curves or surfaces cannot be modified when once their control points are determined. Many authors have studied different kinds of spline for curve and surface with shape parameters through incorporating parameters into the classical basis functions, where the parameters can adjust the shape of the curves and surfaces without changing the control points [7, 8, 9].

Recently, many papers investigate the trigonometric Bézier-like polynomial, trigonometric spline and their applications. Nikolis et al. [10] discussed the applications of trigonometric spline in dynamic systems. Dyllong, Su et al. [11, 12] modified a trajectory for robot manipulators using trigonometric spline. Neamtu et al. [13] presented some methods for designing the cam profile with trigonometric spline in CNC. Many kinds of methods based on trigonometric polynomials were also established for free form curves and surfaces modeling [14, 15, 16, 17, 18, 19]. In recent years, several new trigonometric splines have been studied in the literature; see [20, 21, 22, 23, 24]. Xi-An Han et al. [25] presented the cubic trigonometric Bézier curve with two shape parameters. In this paper the cubic trigonometric polynomial B-spline curve with a shape parameter is presented.

The paper is organized as follows. In section 2, cubic trigonometric B-spline basis functions with a shape parameter are established and the properties of the basis functions are shown. In section 3, cubic trigonometric Polynomial B-spline curves are given and some properties are discussed. By using shape parameter, shape control of the curves is studied. In section 4, the pproximability of the cubic trigonometric Polynomial B-spline curves and cubic Bézier curves corresponding to their control polygons are shown. In section 5, continuity of cubic trigonometric polynomial B-spline curve are discussed. After this some applications and Bicubic Trigonometric B-Spline Surface are also discussed.

CUBIC TRIGONOMETRIC POLYNOMIAL B-SPLINE BASIS FUNCTIONS

Firstly, the definition of cubic trigonometric B-spline basis functions is given as follows.

The Construction of the Basis Functions

Definition: For an arbitrarily selected real values of $-1 \le \lambda \le 1$, $t \in [0, \pi/2]$, the following four functions are defined as cubic trigonometric B-spline basis functions with a shape parameter :

$$\mathbf{B_0(\lambda, t) = f(\lambda)\,(1 - \lambda \sin t)^2\,(1 - \sin t)}$$

$$\mathbf{B_1(\lambda, t) = f(\lambda)\,(1 + \lambda \cos t)^2\,(1 + \cos t)}$$

$$\mathbf{B_2(\lambda, t) = f(\lambda)\,(1 + \lambda \sin t)^2\,(1 + \sin t)}$$

$$\mathbf{B_3(\lambda, t) = f(\lambda)\,(1 - \lambda \cos t)^2\,(1 - \cos t)}$$

Where $\mathbf{f(\lambda)} = \dfrac{1}{4 + 2\lambda^2 + 4\lambda}$

For $\lambda = 0$, the basis functions are general trigonometric polynomials. For $\lambda \neq 0$, the basis functions are cubic trigonometric polynomials.

Fig. 1 shows the curves of the four basis functions

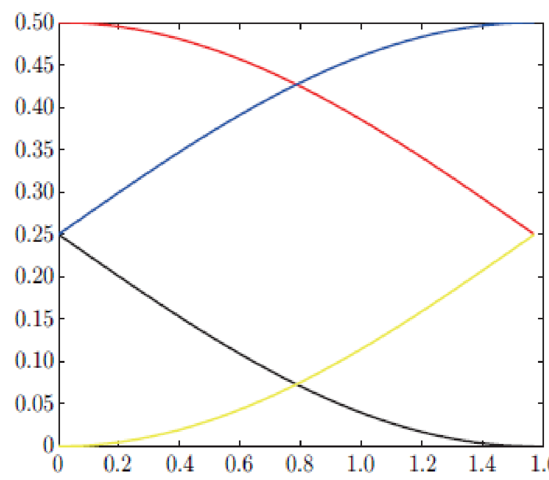

Figure. 1 the basis functions are cubic trigonometric polynomials.

The Properties of the Basis Functions

Theorem 2: T he basis functions (2.1) have the following properties:

Non-Negativity: $B_i(\lambda, t) \geq 0$ for $i \geq 0,1,2,3$
Partition of Unity: $\sum B_i(\lambda, t) = 1$ for $i \geq 0,1,2,3$
Symmetry: $B_0(\lambda, t) = B_3(\lambda, \pi/2 - t)$, $B_1(\lambda, t) = B_2(\lambda, \pi/2 - t)$
Monotonicity for parameter λ : where $t \in [0, \pi/2]$, For a given parameter t, $B_0(\lambda, t)$ And $B_3(\lambda, t)$ are monotonically decreasing for shape parameter λ. $B_1(\lambda, t)$ and $B_2(\lambda, t)$ are monotonically increasing for shape parameter λ . It can be proved by derivation of parameter λ.

CUBIC TRIGONOMETRIC POLYNOMIAL B-SPLINE CURVE

The Construction of the cubic trigonometric Polynomial B-spline curve.

Given points $P_k(k = 0, 1, \cdots, n+1)$ in R^2 or R^3 and knots vectors $U = [u_1, u_2, \cdots, u_n]$, for $i = 1, 2, \cdots, n-1$, then,

$$\mathbf{R_i(t)} = \sum \mathbf{B_j(\lambda, t)\, P_{i+j-1}}, \quad t \in [0, \pi/2] \text{ for } j = 0, 1, 2, 3.$$

The (2) is called cubic trigonometric polynomial B-spline curve segment with a shape parameter.

In the same time, we can give

$$\mathbf{R(u)} = \mathbf{R_i}\left(\frac{\pi}{2} \cdot \frac{\mathbf{u - u_i}}{\Delta \mathbf{u_i}}\right), \quad \mathbf{u \in [u_i, u_{i+1}]}$$

where $\Delta u_i = u_{i+1} - u_i$, $i = 1, 2, \cdots, n-1$, U is equidistant knots vectors, the P (u) is cubic uniform trigono-metric polynomial B-spline curve. Given points $P_0(0, -1)$, $P_1(0.5, 1)$, $P_2(2, 3)$, $P_3(5, 2)$, $P_4(6, 2)$, $P_5(7, 1)$, $P_6(8, -0.5)$, $\lambda = 0.5$, U is equidistant knots vectors,

Figure 2 and 3 shows the cubic trigonometric B-spline curve with different shape parameter.

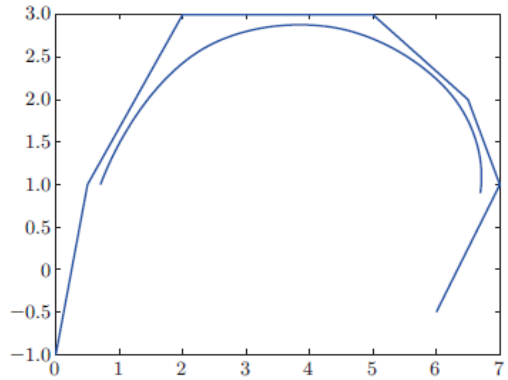

Fig. 2: A cubic trigonometric B-spline curve with parameter $\lambda = 0.2$

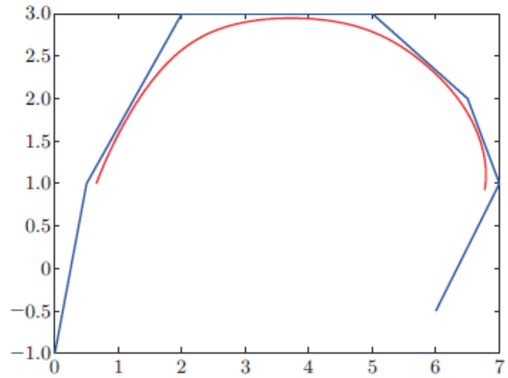

Fig. 3: A cubic trigonometric B-spline curve with parameter $\lambda = 0.5$

From the definition of the basis function some properties of the cubic trigonometric B-spline curve can be obtained as follows:

Theorem 3: The cubic trigonometric Polynomial B-spline curves (3.1) have the following properties:

- **Terminal Properties :**

$\mathbf{R_{i-1}(\pi/2) = R_i(0) = (1/f(\lambda))\,[P_{i-1} + 2(1+\lambda)^2 P_i + P_{i+1}]}$;

$\mathbf{R'_{i-1}(\pi/2) = R'_i(0) = (1/f(\lambda))\,[(-2\lambda - 1)P_{i-1} + (2\lambda + 1)P_{i+1}]}$;

$\mathbf{R''_i(0) = (1/f(\lambda))\,[(2\lambda^2 + 4\lambda)P_{i-1} + (-4\lambda(1+\lambda) - (1+\lambda)^2)P_i + (2\lambda^2 + 4\lambda)P_{i+1} + (1-\lambda)^2 P_{i+2}]}$;

$\mathbf{R''_i(\pi/2) = (1/f(\lambda))[(1-\lambda)^2 P_{i-1} + (2\lambda^2 + 4\lambda)P_i + (-4\lambda(1+\lambda) - (1+\lambda)^2)P_{i+1} + (2\lambda^2 + 4\lambda)\,P_{i+2}]}$;

- **Symmetry** : P_0, P_1, P_2, P_3 and P_3, P_2, P_1, P_0 define the same cubic trigonometric Polynomial B-spline curve in different parameterizations, i.e.,

  $$\mathbf{R(t; \lambda; P_0, P_1, P_2, P_3) = R(1- t; \lambda; P_3, P_2, P_1, P_0) ; t \in [0, \pi/2] , \lambda \in [-1,+1]}$$

- **Geometric Invariance:** The shape of a cubic trigonometric Polynomial B-spline curve is independent of the choice of coordinates, i.e. (3.1) satisfies the following two equations:

$$\mathbf{R(t; \lambda; P_0 + q , P_1 +q, P_2 +q, ,P_3 +q) = R(1- t; \lambda; P_3, P_2, P_1, P_0) + q ;}$$
$$\mathbf{R(t; \lambda; P_0 *T , P_1 *T , P_2 *T , ,P_3 *T) = R(1- t; \lambda; P_3, P_2, P_1, P_0) * T ;}$$

Where q is arbitrary vector in R^2 or R^3 and T is an arbitrary d * d matrix, d = 2 or 3.

- **Convex Hull Property:** The entire cubic trigonometric Polynomial B-spline curve segment lies inside its control polygon spanned by P_0,P_1,P_2,P_3.

Shape Control of the Cubic Trigonometric Polynomial B-Spline Curve

For $t \in [0, \pi/2]$, $\lambda \in [-1,+1]$, Given points $P_k(k = 0, 1, \cdots, n+1)$ in R^2 or R^3 and knots vectors $U = [u_1,u_2, \cdot\cdot\cdot , u_n]$, for i = 1, 2, · · · , n −1, then, we rewrite (3.1) as follows:

$$\mathbf{R_1(t)} = \sum_{i=0}^{3} P_i C_i(t) + \lambda^2 (P_0 + P_2)\sin^2 t + \lambda^2 (P_2 - P_0)\sin^3 t + \lambda^2 (P_1 + P_3)\cos^2 t + \lambda^2 (P_1 - P_3)\cos^3 t$$
$$+2\lambda (P_2 - P_0)\sin t + 2\lambda (P_2 + P_0)\sin^2 t + 2\lambda (P_1 - P_3)\cos t + 2\lambda (P_1 + P_3)\cos^2 t$$

where

$$C_0(t) = (1 - \sin t), C_1(t) = (1 + \cos t),$$
$$C_2(t) = (1 + \sin t), C_3(t) = (1 - \cos t)$$

Obviously, shape parameter affects curve on the control edge ($P_0 + P_2$),($P_2 - P_0$), ($P_1 - P_3$),and ($P_1 + P_3$),. Therefore as the shape parameter increases, the cubic trigonometric Polynomial B-spline curve approximates the control polygon. Figure 4 shows some computed examples with different values of shape parameters.

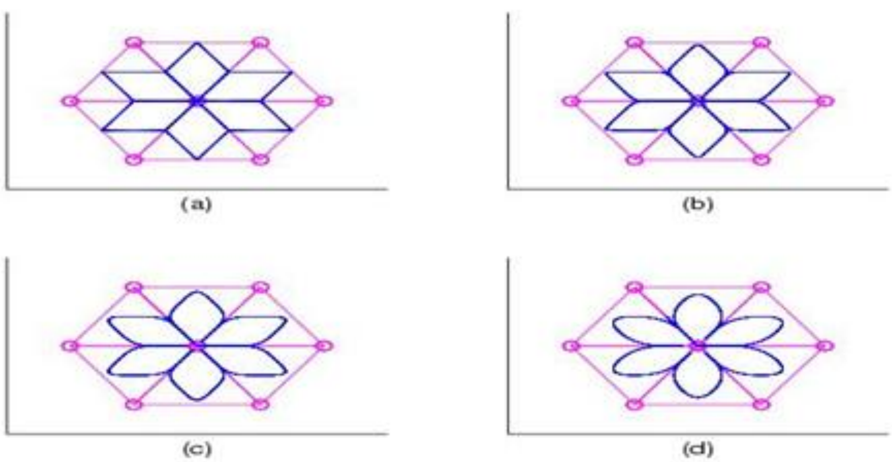

Figure 4: Cubic Trigonometric Polynomial B-Spline Curves with Different Values of Shape Parameter.

These curves are generated by setting λ =1 in (a), λ =0 in (b), λ =-0.5 in (c) and λ =-1 in (d).

APPROXIMABILITY

Control polygon provides an important tool in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relations of the cubic trigonometric Polynomial B-spline curve and cubic Bézier curves corresponding to their control polygon.

Control polygon provides an important tool in geometric modeling. It is an advantage if the curve being modeled tends to preserve the shape of its control polygon. Now we show the relations of the cubic trigonometric Polynomial B-spline curve and cubic Bézier curves corresponding to their control polygon.

Theorem 4: Suppose P_0,P_1,P_2,and P_3 are not collinear; the relationship between cubic trigonometric Polynomial B-spline curve $\mathbf{R_i}$ **(t)** and the cubic Bézier curve $B_i(t) = \sum_{j=0}^{3} p_{i+j-1}(3,j)(1-t)^{3-j}t^j$; $t \in [0, 1]$ with the same control points P_i (i=0,1,2,3) are as follows**:**

$R_i(0) = (1/f(\lambda))\,[Pi - 1 + 2(1+\lambda)^2 Pi + Pi+1]$; $B_i(0) = P_{i-1}$;

$R_i(\pi/2) = (1/f(\lambda))\,[Pi+1 + 2(1+\lambda)^2 Pi+2 + Pi+3]$; $B_i(1) = P_{i+2}$;

$$R_i(\pi/4) = \frac{f(\lambda)}{2\sqrt{2}}\left[\left(\sqrt{2}-\lambda\right)^2\left(\sqrt{2}-1\right)\left(8\left(B_i\left(\frac{1}{2}-P^*\right)\right)+P_{i-1}+P_{i+2}\right)+\left(\sqrt{2}+\lambda\right)^2\left(\sqrt{2}+1\right)\left(P_i+P_{i+1}\right)\right]$$

Where $P^* = P_{i+1} + P_{i+2}$

Proof: According to (3.2), we have

$R_i(0) = (1/f(\lambda))\,[Pi - 1 + 2(1+\lambda)^2 Pi + Pi+1]$;

and

$R_i(\pi/2) = (1/f(\lambda))\,[Pi+1 + 2(1+\lambda)^2 Pi+2 + Pi+3]$;

Performing simple computation, we have

$B_i(0) = P_{i-1}$; and $B_i(1) = P_{i+2}$;

$B_i(1/2) - P^* = (1/8)\,(P_{i-1} - P_i - P_{i+1} + P_{i+2})$

and according to (4.2), we have

$$R_i(\pi/4) = \frac{f(\lambda)}{2\sqrt{2}}\left[\left(\sqrt{2}-\lambda\right)^2\left(\sqrt{2}-1\right)\left(P_{i-1}+P_{i+2}\right)+\left(\sqrt{2}+\lambda\right)^2\left(\sqrt{2}+1\right)\left(P_i+P_{i+1}\right)\right]$$

$$= \frac{f(\lambda)}{2\sqrt{2}}\left[\left(\sqrt{2}-\lambda\right)^2\left(\sqrt{2}-1\right)\left(8\left(B_i\left(\frac{1}{2}-P^*\right)\right)+P_{i-1}+P_{i+2}\right)+\left(\sqrt{2}+\lambda\right)^2\left(\sqrt{2}+1\right)\left(P_i+P_{i+1}\right)\right]$$

Then (4.1) holds.

From Figure 5, we can see that the cubic trigonometric Polynomial B-spline curve (red lines for λ=1, λ=0 and λ= -1 respectively) is closer to the control polygon than the cubic Bézier curve (blue lines) for all values of $\boldsymbol{\lambda \in [-1,+1]}$.

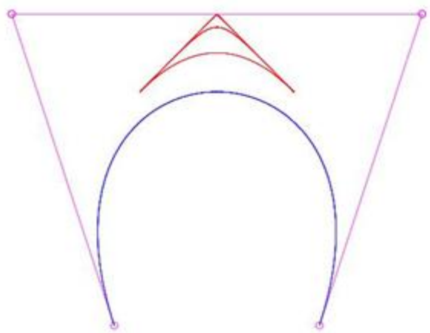

Figure 5: The Relationship between Cubic trigonometric Polynomial B-spline Curve and the Cubic Bézier Curves.

CONTINUITY

The (2) is analogous to the represent of cubic B-spline curve segment, so it has same properties with cubic B-spline curve: geometric invariance, convex hull property, symmetry and locality so on.

Theorem 2: For cubic trigonometric polynomial B-spline curve (3), its continuity is as follows

$$\mathbf{R}_i^{(K)}(u_i-) = \left(\frac{\Delta u_i}{\Delta u_{i-1}}\right)^K \mathbf{R}_i^{(K)}(u_i+)$$

when λ = 1, k = 0, 1, 2, 3; (b) when λ≠ 1, k=0, 1, 3.

Proof: For (2), according to simple differential operation, until to third derivation, and more calculate, we can give:

$R_{i-1}(\pi/2) = R_i(0) = (1/f(\lambda))\,[P_{i-1} + 2(1+\lambda)^2 P_i + P_{i+1}]$;

$R'_{i-1}(\pi/2) = R'_i(0) = (1/f(\lambda))\,[(-2\lambda-1)P_{i-1} + (2\lambda+1)P_{i+1}]$;

$R''_i(0) = (1/f(\lambda))\,[(2\lambda^2+4\lambda)P_{i-1} + (-4\lambda(1+\lambda)-(1+\lambda)^2)P_i + (2\lambda^2+4\lambda)P_{i+1} + (1-\lambda)^2 P_{i+2}]$;

$R''_i(\pi/2) = (1/f(\lambda))[(1-\lambda)^2 P_{i-1} + (2\lambda^2+4\lambda)P_i + (-4\lambda(1+\lambda)-(1+\lambda)^2)P_{i+1} + (2\lambda^2+4\lambda)P_{i+2}]$;

$\mathbf{R}^{(K)}_{i-1}(\pi/2) = \mathbf{R}^{(K)}_{i}(0) = (1/f(\lambda))[(-6\lambda^2+2\lambda+1)P_{i-1} + (6\lambda^2-2\lambda-1)P_{i+1}]$.

so that, $\mathbf{R}^{(K)}_{i-1}(\pi/2) = \mathbf{R}^{(K)}_{i}(0)$ where λ≠1 and k = 0, 1, 3.

Let $u \in [u_i, u_{i+1}]$; $t = \frac{\pi}{2} * \frac{u - u_i}{\Delta u_i}$,

then

$$\mathbf{R}^{(K)}(u) = \left(\frac{\pi}{2} * \frac{1}{\Delta u_i}\right)^K \mathbf{R}^{(K)}_{i}(t);$$

then

$$(\mathbf{R}^{(K)}(u_i-) = \left(\frac{\pi}{2} * \frac{1}{\Delta u_{i-1}}\right)^K \mathbf{R}^{(K)}_{i-1}\left(\frac{\pi}{2}\right); \text{ and } \mathbf{R}^{(K)}(u_i+) = \left(\frac{\pi}{2} * \frac{1}{\Delta u_i}\right)^K \mathbf{R}^{(K)}_{i-1}(0)$$

According to above two equations, the Theorem 2 holds. The Theorem 2 shows that $R_i(u)$ is C^1 continuity if $\lambda \neq 1$, and else $\lambda = 1$, is C^3 continuity.

APPLICATIONS

The construction of open curve and closed curve is the most basic content of curve design; people should know terminal behaviors of the open curve and how to construct a closed curve. Given closed control points $P_i (i = 0, 1, \cdots, n)$, where $P_n = P_0$, if $P_{n+1} = P_1$ and $P_{n+2} = P_2$, then we can construct an closed trigonometric polynomial B-spline curve; if there are open control points $P_i (i = 0, 1, \cdots, n)$, where $P_0 = 2P_1 - P_2$, $P_{n+1} = 2P_n - P_{n-1}$, we can construct an open trigonometric polynomial B-spline curve, both sides of which are interpolated in P_1 and P_n, also tangential vectors of knots u_1 and u_n are P_2-P_1 and P_n-P_{n-1}, respectively.

BICUBIC TRIGONOMETRIC B-SPLINE SURFACE

Definition 3 Given (m+1)×(n+1) control points $P_{kl} (k = 0, 1, \cdots m+1; m = 0, 1, \cdots, n+1)$, two knot vectors $U = [u_1, u_2, \cdots, u_m]$ and $V = [v_1, v_2, \cdots, v_n]$, for $i = 1, 2, \cdots, m-1$, $j = 1, 2, \cdots, n-1$, We define bicubic trigonometric B-spline surface patch in the following, using tensor product:

$$r_{ij}(u,v) = \sum_{k=0}^{3}\sum_{m=0}^{3} P_{i+k-1,j+m-1} B_k(\lambda,u) B_m(\lambda,v); 0 \le u,v \le \frac{\pi}{2}$$

Then the bicubic trigonometric B-spline surface

$$r(u,v) = r_{ij}\left(\frac{\pi}{2} * \frac{u-u_i}{\Delta u_i}, \frac{\pi}{2} * \frac{v-v_j}{\Delta v_j}\right); u \in [u_i, u_{i+1}]; v \in [v_j, v_{j+1}]$$

where $\Delta u_i = u_{i+1} - u_i$, $i = 1, 2, \cdots, m-1$; *and* $\Delta v_j = v_{j+1} - v_j$, $j = 1, 2, \cdots, n-1$.

Clearly, the surface has similar properties with the curve. Where $\lambda = 1$, uniform cubic trigonometric B-spline surface is C^3continuous, and has a better approximation. As with one parameter, we can adjust shape of the surface, which is more confident to design shape. Fig. 8 shows bicubic trigonometric B-spline surface patch with different parameters $\lambda = 1$ (red) and $\lambda = 0$ (blue).

Conclusions

This work defines a cubic trigonometric B-spline curve with one parameter, and analysis properties of its basis functions. Each section of the curve only refers to four control points. We can design different shape curves by changing parameter. Where $\lambda = 1$, uniform cubic trigonometric B-spline curve is $C3$ continuous. The curve can represent ellipse when adjusting the control points and parameter value. We can design open curve and closed curve by using the properties of multipleknots. We put B-spline curve forward to bi-cubic B-spline surfaces using tensor product.

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