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Alternative Approximations to Cobb-Douglas Production Function: A Revisit

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Abstract
This paper shows that a bivariate Taylor series approximation of Cobb-Douglas (CD) production function fares well in practice. A limitation of well-known Cobb-Douglas production function can be tackled with such linear and quadratic Taylor approximations. However this comes at a price of violation of some fundamental assumptions in CD model. Regression analysis reveals that Linear Taylor approximation (TAL) and Quadratic Taylor approximation (TAQ) have some flexibility over Cobb-Douglas way of characterizing the production function.

Keywords: Cobb-Douglas (CD), CRS, Taylor series, Taylor approximation, marginal productivity, elasticity

Introduction
Models play a major role in most economic studies, no matter whether theoretical or empirical. There are alternatives specifications of models. An estimated model under a suitable specification can then be used for various purposes, including structural analysis, forecasting and policy evaluation. In economics, Cobb-Douglas (CD) form of production function is widely used to represent the relationship of an output to different input factors. It was proposed by Wicksell (1851-1926) and tested against statistical evidence by Cobb and Douglas (1928).

In 1920s the economist Paul Douglas's work related to the input and output at the national aggregate level. Mathematician Charles Cobb determined a proper relationship between the capital-labor factor and production; giving birth to the original production function. They simplified the view of the economists in which production output is determined by the amount of labor involved and the capital invested; though there are many other factors affecting economic performance. Their work basically modeled the growth of American economy during the period 1899-1922 (table-05), "A theory of production" (Cobb and Douglas, 1928). In this article we investigate whether there is any other way to demonstrate the relationship between input-output similar to what Cobb-Douglas did. We observed that Taylor approximations, even linear one, can provide competitive alternatives to Cobb-Douglas's model. In fact interpolating slow growth nature and considering the flexibility in functional form we found that the linear

Taylor approximation is a very convenient model for production prediction.

1. COBB-DOUGLAS PRODUCTION FUNCTION

Let, the formula $P = P(L, K)$ represent the relationship between output P-the production, labor L and capital input K. Assume that $P(L,K)$ is continuously differentiable. For the production function $P = P(L, K)$, the partial derivative $\frac{\partial P}{\partial L}$ is the rate at which the production changes with respect to the unit change of labor. Economists call it the *marginal productivity of labor*. Likewise, $\frac{\partial P}{\partial K}$ is the *marginal productivity of capital*.

1.1 Cobb-Douglas assumption

The assumptions made by Cobb and Douglas can be stated as follows:

a) If either labor or capital vanishes, then so will production.
b) The marginal productivity of labor is proportional to the amount of production per unit of labor.
c) The marginal productivity of capital is proportional to the amount of production per unit of capital.

1.2 Derivation of the Model

Under the assumption (b) we can say that $\frac{\partial P}{\partial L} = \alpha_1 \frac{P}{L}$, for some constant α_1.
Then by considering K as constant and integrating with respect to L, we get

$$\ln P = \alpha_1 \ln L + g(K) + c_1 \quad (1.1)$$

Similarly, assumption (c) say that $\frac{\partial P}{\partial K} = \beta_1 \frac{P}{K}$, for some constant β_1, and we get

$$\ln P = \beta_1 \ln K + g(L) + c_2 \quad (1.2)$$

Combining equation (1.1) and (1.2), we get

$\ln P = \alpha \ln L + \beta \ln K + C$, for some constants α and β.

Which implies $$P(L,K) = AL^{\alpha} K^{\beta}$$
(here $C = \ln A$) (1.3)

This (1.3) is the *Generalized Cobb-Douglas Production Model*, we write it as *CD-Gen.*
In (1.3) *A* is a constant called the *total factor productivity* and independent of *L* and *K*.
The constants α and β are the *output elasticity* of labor and capital respectively. According to the CD assumptions the parameters of the models are positive, i.e. $\alpha > 0$, $\beta > 0$. These constants can be determined by using available technologies.

If labor L and capital K both are increased by a factor m, then we get from (1.3) that

$$\begin{aligned} P(mL, mK) &= A(mL)^{\alpha} (mK)^{\beta} \\ &= Am^{\alpha+\beta} L^{\alpha} K^{\beta} \\ &= m^{\alpha+\beta} P(L,K) \end{aligned} \quad (1.4)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L,K)$, which means that production is also increased by a factor of *m*.
According to the estimated numerical values of α and β, the production model $P(L,K)$ differentiates as follows:

- If $\alpha + \beta = 1$, we say that *P* has *constant returns to scale* (CRS).
- If $\alpha + \beta > 1$, we say that *P* has *increasing returns to scale*(IRS).
- If $\alpha + \beta < 1$, we say that *P* has *decreasing returns to scale* (DRS).

1.3 The Cobb Douglas CRS Model

If in the generalized Cobb-Douglas production model, it has been assumed that $\alpha+\beta=1$, then the model (1.3) is written as:

$$P = AL^{\alpha} K^{1-\alpha} \quad (1.5)$$

which we call *Cobb-Douglas model for Constant Returns to Scale (CD-CRS).*

The log-linear form of (CRS) Cobb-Douglas production function (1.5) is:

$$P = AL^{\alpha} K^{1-\alpha}$$

$$\ln(P) = \ln A + \alpha \ln(L) + (1-\alpha)\ln K \quad (1.6)$$

$$\ln\left(\frac{P}{K}\right) = \ln A + \alpha \ln\left(\frac{L}{K}\right)$$

Replacing $\ln\left(\frac{P}{K}\right)$ by z, $\ln\left(\frac{L}{K}\right)$ by x and $\ln A$ by a, we get the linear equation in the parameters a and α as:

$$z = a + \alpha x \quad (1.7)$$

2. OBJECTIVE OF THE WORK

The Cobb-Douglas production function is often seen as panacea for analyzing production process or is dubbed as a simplistic tool based on some restrictive conditions. The conditions imposed by a restricted CD function $P = AL^{\alpha}K^{\beta}$ are (i) $\alpha, \beta > 0$ (ii) $\alpha+\beta=1$ (iii) $P_L, P_K > 0$ (iv) $L, K > 0$. Under the last assumption- $L, K > 0$, the rest of the conditions are not necessarily be followed by the empirical model fitting through available statistical techniques. The common criticism about the model is that it is inflexible in its functional form; e.g. the unitary elasticity condition of the model which is naturally unrealistic. Moreover a production function depending on very high number of input-factors is difficult to handle by CD-model. The inflexibility comes from the fact that the CD production function is a non-linear model for empirical study with limited number of observations. That is why the linearity assumption has frequently been investigated first for economic models. Because of the lack of the flexibility in evolution, the non-linear exponential or transcendental functions are often approximated by Taylor theorem (Taylor, 1715). In the model $P = AL^{\alpha}K^{\beta}$, A is considered as independent of L and K. But it is not all that obvious how to estimate 'A' from a set of time series data. In practice regression based statistical estimation doesn't turn out to be handy for this form of CD model.

In fact, in the model $P = AL^{\alpha}K^{\beta}$, if either L or K is zero, then the product is zero, which is not realistic. Also its log linear form $\ln P = C + \alpha \ln L + \beta \ln K$ is undefined at any zero level of input or output factors. Due to these inconsistences of CD model the Taylor series approximation appears as a convenient alternative for the estimation of CD model.

3. Taylor Series Approximation of Cobb-Douglas Model

For a function f that depends on two variables, *x* and *y*, the Taylor series expansion of $f(x, y)$ about the point (a, b) up to third order is:

$$f(x, y) \approx f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(x-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a^3)\frac{\partial^3 f}{\partial x^3}(a,b) + 3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2 \partial y}(a,b) + 3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x \partial y^2} + (y-b)^3\frac{\partial^3 f}{\partial y^3}(a,b)\right] + \dots \dots \dots \quad (3.1)$$

Any smooth function can be reasonably approximated in linear or quadratic fashion using Taylor's theorem. Consider for example, the general production function:

$$P = F(L,\ K)$$

of which the generalized Cobb-Douglas form (1.3) is one special case. If the function is continuous and n-times differentiable, then it can be approximated as a series by taking (n+1) terms of the Taylor series expansion. Expanding the CD function $F(L,K)$ about the base levels of $(K_0,\ L_0)$ as (1.2), we get

$$\tilde{P} = F(L,K) \approx F(L_0,K_0) + (L-L_0)\frac{\partial F}{\partial L}(L_0,K_0) + (K-K_0)\frac{\partial F}{\partial K}(L_0,K_0)$$
$$+\frac{1}{2!}\left[(L-L_0)^2\frac{\partial^2 F}{\partial L^2}(L_0,K_0) + (K-K_0)^2\frac{\partial^2 F}{\partial K^2}(L_0,K_0) + 2(L-L_0)(K-K_0)\frac{\partial^2 F}{\partial L \partial K}(L_0,K_0)\right] + \dots \dots \dots \quad (3.2)$$

This is the second order Taylor series expansion of the production function $F(L,\ K)$.

3.1 Linear Approximation of CD

Taking the constant and linear terms in L and K from (3.2), we get

$$\tilde{P} = F(L_0,K_0) + (L-L_0)\frac{\partial F}{\partial L}(L_0,K_0) + (K-K_0)\frac{\partial F}{\partial K}(L_0,K_0) \quad (3.3)$$

Then

$$\tilde{P} = a + \alpha K + \beta L \quad (3.4)$$

where the parameters can be interpreted as

$$a = F(L_0,K_0) - L_0\frac{\partial F}{\partial L}(L_0,K_0) - K_0\frac{\partial F}{\partial K}(L_0,K_0)$$

$$\alpha = \frac{\partial F}{\partial L}(K_0,\ L_0) \quad \text{and} \quad \beta = \frac{\partial F}{\partial K}(K_0,\ L_0)$$

Under the Cobb-Douglas assumption $P = 0$ for $L = K = 0$, that is $F(0,0) = 0$, then we get $a = 0$, and the linearly approximated model (3.4) under this assumption is:

$$\tilde{P} = \alpha L + \beta K \quad (3.5)$$

Which is the *Taylor Approximation to Linear*, denoted as *(TAL)*, of the production function in the variables L and K.

3.2 Quadratic Approximation of CD

Taking term up to second order of (3.2), considering the Cobb-Douglas assumption $P = 0$ for $L = K = 0$, and simplifying as (3.3) to (3.4), we get

$$\tilde{P} = \alpha L + \beta K + \gamma L^2 + \lambda K^2 + \delta LK \quad (3.6)$$

Which is the *Taylor Approximation to Quadratic*, denoted as *(TAQ)*, of the production function in variables L and K.

4. THEORITICAL COMPARISON

The CDs lacks flexibility in functional form; e.g. it needs to be log-linearized in parameters though the real models are non-linear in variables. No well accepted methodology is established with the theory of CD functions to determine the parameter values and their limit. In contrast the linear and quadratic Taylor approximations TAL (3.5) and TAQ(3.6) are more manageable and flexible in functional form and are more handy in empirical estimation with statistical regression. Moreover the Taylor approximations are free from the independent constant term, which are included in CDs. Zero level of input of any variable in CDs the model output become meaningless and also its log-linear form is undefined at this level, but the Taylor approximations are not so for this kind of input level. If $P_1 = AL_1^{\alpha} K_1^{\beta}$ and $P_2 = AL_2^{\alpha} K_2^{\beta}$, then in general, $P_1 + P_2 = A(L_1 + L_2)^{\alpha} (K_1 + K_2)^{\beta}$ is not analytically convenient though it should be. These types of typical mathematical manipulations are convenient in Taylor series approximations, for example if $P_1 = \alpha L_1 + \beta K_1$ and $P_2 = \alpha L_2 + \beta K_2$ then $P_1 + P_2 = \alpha(L_1 + L_2) + \beta(K_1 + K_2)$.

5. Numerical Analysis of the Model Efficiency

The efficiency of the models imply the goodness of fit and power of prediction of the estimated models outputs compared to the real observed outputs. The efficiency of the model is analyzed by observing the simulated numerical results, their simultaneous geometrical figures and the statistical evidence.

Table 1: The estimated output for generalized Cobb-Douglas (CD-Gen), CRS-Cobb-Douglas (CD-CRS), Taylor Series Approximation to Linear (TAL), and Taylor Series Approximation to Quadratic (TAQ) are presented together with the real observed value for the corresponding labor-capital inputs.

Time	Real	Input		Estimated Value: $\tilde{P}$			
Year	P	L	K	CD-CRS	CD-Gen	TAL	TAQ
1955	114043	8310	182113	116413.0	115932.9	119793.1	109522.7
1956	120410	8529	193749	123919.2	123253.8	127096.8	118765.0
1957	129187	8738	205192	131304.7	130453.2	134272.2	127728.4
1958	134705	8952	215130	137712.8	136899.6	140540.3	135502.4
1959	139960	9171	225021	144090.2	143378.7	146785.6	143180.3
1960	150511	9569	237026	151799.5	152003.2	154514.2	152835.7
1961	157897	9527	248897	159532.1	158180.8	161667.2	160923.5
1962	165286	9662	260661	167154.4	165270.8	168954.2	169405.8
1963	178491	10334	275466	176616.0	177182.8	178688.8	181383.4
1964	199457	10981	295378	189409.0	191879.7	191493.0	196905.3
1965	212323	11746	315715	202447.5	207699.7	204687.5	213574.8
1966	226977	11521	337642	216794.6	218395.3	217734.3	224763.8
1967	241194	11540	363599	233720.1	232644.6	233499.6	240216.1
1968	260881	12066	391847	251994.4	251629.1	251224.0	258111.9
1969	277498	12297	422382	271864.5	269852.2	270004.1	276027.9
1970	296530	12955	455049	292990.7	292543.6	290557.1	295065.4
1971	306712	13338	484677	312229.3	311648.3	308957.8	311262.9
1972	329030	13738	520553	335555.5	334395.4	331167.3	329989.3
1973	354057	15924	561531	361572.5	374893.8	358477.6	355077.2
1974	374977	14154	609825	393874.9	386196.0	385781.6	375229.1

Note: The left side boxes show the real input-output relations. The source of the data is the Mexican Economy over the period 1955-1974.

In Table 01, it is observed that the labor inputs are very small compared to the capital inputs. The estimated values $\tilde{P}$, given in the table are the respective outputs of the estimated models. Here, the models for prediction were estimated on whole of the population. Comparing with the real output we can say all estimated models are fitted very well within the population. In the real observed outputs, they are monotonic increasing over time and this phenomena are observed by all the predicted models. These are evident in table-01 and figure-01.

In figure-01, the estimated model outputs and the real observed values are presented. All the models are potential enough in prediction. From the upper end of the presented graphs it is our observation that the efficiency of Taylor approximations is better than that of Cobb-Douglas.

In Table 02 the estimated models are given. Observing the values of r^2 in this table, clearly all models are numerically well fitted within the population. But criticisms over CD model are that the values of α and β do not have nearly unique approximation over the population and sample. It is evident from the estimated CD models that the values α and β have shown the rapid change in values depending on the population or on sample size. Even their values may significantly change for the change of a few numbers of the observations. However in most cases model fits go well. When it was forced to the CD model to have CRS, then from the second row of this table we see that $\alpha < 0$ which is a clear violation of CD assumption. This violation of assumption does not weaken the model fitness as predictor. In this table the corresponding numerical estimated models of TAL and TAQ, and their respective values of r^2 are given. According to this statistical tool, clearly TAL and TAQ are also well fitted models.

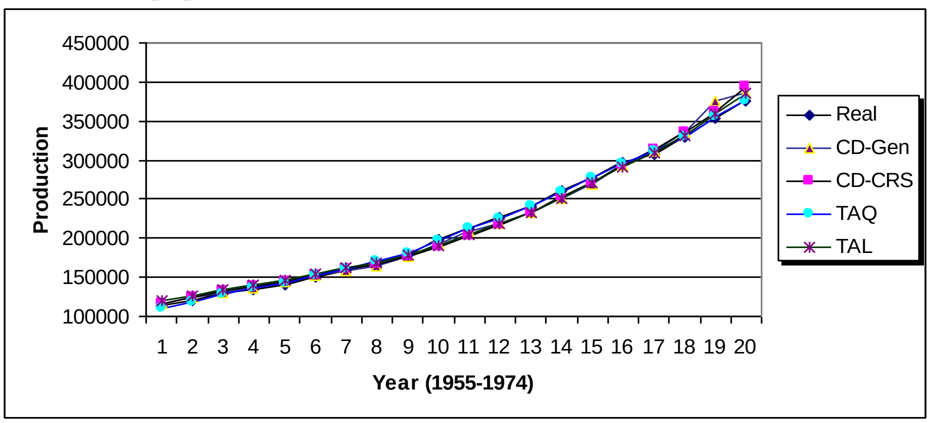

Fig. 1: Estimated values vs real observed values of production function over the period 1955 to 1974.

Table 2: Numerical models for estimation for whole period (1955 to 1974)

Model Name	Numerical model for estimation	r^2
CD-Gen	$\tilde{P} = 0.191308\,L^{0.340155}\,K^{0.845801}$	0.992036
CD-CRS	$\tilde{P} = 0.609739\,L^{-0.0153024}\,K^{1.0153024}$	0.992072
TAL	$\tilde{P} = 1.12328\,L + 0.606539\,K$	0.999239
TAQ	$\tilde{P} = -21.531\,L + 1.326876\,K + 0.002316\,L^2 + 2.85\times10^{-7}K^2 - 8.1\times10^{-5}\,L\,K$	0.999887

Table 3: Estimated results by using Numerical models over the period 1955 to 1971.

Model	Numerical Model for Estimation	r^2 within Sample	Pred. Err.
CD-Gen	$\tilde{P} = 0.0419264\,L^{0.648561}\,K^{0.740029}$	0.9953489	5818.1
CD-CRS	$\tilde{P} = 0.463794\,L^{-0.0993945}\,K^{1.0993945}$	0.9929040	4845.8
TAL	$\tilde{P} = -0.82257L + 0.677832K$	0.999366	3605.5
TAQ	$\tilde{P} = 25.83141L - 0.46125K - 0.00601L^2 - 4.6\times10^{-6}K^2 + 0.000363\,L\,K$	0.999899	11145.7

In table-03, estimated models are presented which are derived on the sample observations over 1955-1971. With respect to the value of r^2, all models are fitted well within the sample, and Taylor approximations are slightly better than that of CDs. The prediction error (3605.5) of TAL is less than any other models. So it can be commented that the power of TAL model is more than any other derived models. In the CD-Gen model the sum of the parameters α and β are not approximately equal to 1. Moreover we observed significant changed in the parameters A, α and β when only three observations (in 1972-1974) were omitted to estimate the numerical models within the sample. Here α changes from 0.648561 to 0.340155. In terms of numerical output and statistical measurement the CD-CRS model fitted fine.

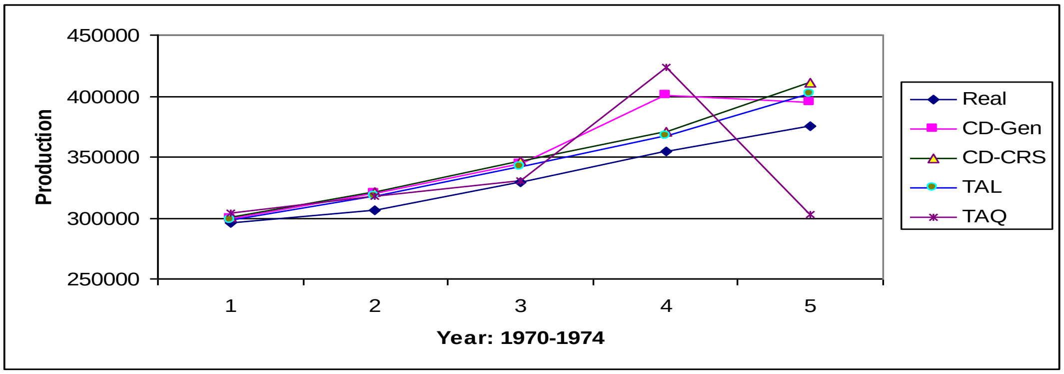

Fig. 2: Estimated values vs. real observed values of production function over the period 1970 to 1974.

Figure-02 is the graphical representations of the estimated models output together with the real observation within the sample (1970-1971) and outside the sample (1972-1974). The empirical outputs corresponding to the figure-02 are given in table-04. Here, it is observed that TAL is fitted better and has consistent growth outside the sample. Whereas the growth of CD-Gen and TAQ are not consistent in characteristics outside the sample compared to real outputs. Here, it is clear that the efficiency of the TAL is more powerful than the others.

In Table-04, few of the output of the models (shown in table-03) are given. The models were estimated for 1955-1971 and used them to future predictions on 1972-1974. The predicted values outside the sample are shown in the shaded boxes of the table.

Table 4: Predicted values from 1972 to 1974 from the observed values 1955-1971.

Year	Real: P	L	K	CD-Gen	CD-CRS	TAL	TAQ
1970	296530	12955	455049	299765.2	300420.4	297790.4	303507
1971	306712	13338	484677	320081.2	321060.9	317558.1	317853.1
1972	329030	13738	520553	343980.0	346262.5	341547	329939.3
1973	354057	15924	561531	400385.2	370859.6	367525.1	423776.8
1974	374977	14154	609825	394280.7	410853.5	401716.2	302861.4

Table 5: Estimated and observed labor, capital & production function for U.S. national economy index over the period 1899 to 1922.

Time	Output	Input		Estimated Value: $\tilde{P}$			
Year	Real: $\tilde{P}$	L	K	CD-Gen	CD-CRS	TAL	TAQ
1899	100	100	100	100.44	100.71	102.58	99.80
1900	101	105	107	106.04	106.25	108.05	106.06
1901	112	110	114	111.63	111.79	113.52	112.38
1902	122	117	122	119.03	119.09	120.87	121.16
1903	124	122	131	125.11	125.12	126.69	127.76
1904	122	121	138	125.93	126.02	127.04	127.12
1905	143	125	149	131.58	131.66	132.35	132.65
1906	152	134	163	141.92	141.87	142.44	144.24
1907	151	140	176	149.60	149.48	149.80	152.24
1908	126	123	185	137.10	137.48	136.84	134.29
1909	155	143	198	156.54	156.49	156.15	157.17
1910	159	147	208	161.86	161.76	161.28	162.49
1911	153	148	216	164.23	164.16	163.52	164.19
1912	177	155	226	172.08	171.88	171.21	172.95
1913	184	156	236	174.80	174.62	173.79	174.72
1914	169	152	244	172.76	172.74	171.75	170.88
1915	189	156	266	180.04	180.04	178.95	177.19
1916	225	183	298	209.35	208.73	207.51	209.49
1917	227	198	335	228.95	228.06	226.68	228.60
1918	223	201	366	236.73	235.90	234.58	233.14
1919	218	196	387	235.41	234.84	233.92	229.25
1920	231	194	407	236.49	236.07	235.66	229.45
1921	179	146	417	191.21	192.23	196.41	201.41
1922	240	161	431	207.83	208.50	211.62	211.69

The data source of the following table is in the index form of the national economy 1899-1922 of United States, where 1899 were as the base year, and a standard level of 100 units for labor, capital, and production for that year.

In the original source of data some irregularities are seen, for example comparing years 1902 and 1904, their production levels were same though their input level in labor and capital were different in units. This phenomena are also seen in year 1911, 1917 etc. This kind of irregularities may be usual in real life. But CD in its own characteristics for $\alpha \cdot \beta > 0$ is a monotonic function. So CD function is not capable to follow this kind of characteristics seen in the real observed data. The other models- TAL and TAQ are not free from these types of theoretical limitations. But difference is that the linear or polynomial functions are slower in growth or decay than exponential functions. So in these cases, linear or quadratic approximations may be more convenient than that of CDs. According to the numerical output of the four estimated models shown in table-05, often their predicted outputs are very close to each other though they are not closer to the given real observations. From the observed numerical evidence for different input factors, the predicted models are almost similar in prediction efficiency.

The following numerical models are estimated by taking the whole observations from 1899-1922.

In the table-06, estimated CDs models, both CD-Gen and CD-CRS satisfy the CD conditions $\alpha, \beta > 0$ and even in CD-Gen $\alpha + \beta \approx 1$. According to the calculated value of r^2, the Taylor approximations are fitted better than that of CDs.

Fig 03 shows the year by year line graphs for the predicted model output together with the real observed values. The CD model fails to satisfy the actual oscillatory behavior of real outputs. This limitation is also true for the Taylor approximations. But comparing the graphical representations and r^2 value, it can be commented on the models that the Taylor approximations are more powerful than that of the CDs within the population.

Table 6: Estimated results by using Numerical models over the period 1899 to 1922.

Model	Numerical model for estimation	r^2
CD-Gen	$\tilde{P} = 0.933131 L^{0.768866} K^{0.247111}$	0.955017
CD-CRS	$\tilde{P} = 1.00707 L^{0.744606} K^{0.255394}$	0.939343
TAL	$\tilde{P} = 0.853625 L + 0.172125 K$	0.996306
TAQ	$\tilde{P} = 0.416001 L + 0.380747 K + 0.00522 L^2 + 0.000583 K^2 - 0.00379 L K$	0.996495

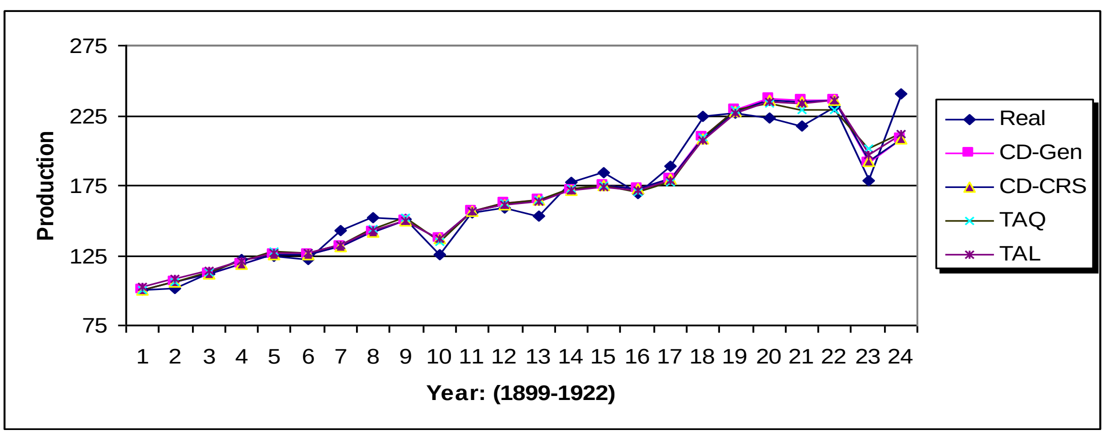

Fig 3: Point to point graph of observed values together with estimated values.

Table 7: Numerical models estimated by considering the given output from 1899 to 1919 as sample to observe the efficiency of the models within the sample 1899-1919 as well as outside the sample 1920-1922.

Model	Numerical model for estimation	r^2	Pred. Err
CD-Gen	$\tilde{P} = 0.365185 L^{1.201113} K^{0.017984}$	0.96802	6.71
CD-CRS	$\tilde{P} = 1.01667 L^{0.78291} K^{0.21709}$	0.93577	4.23
TAL	$\tilde{P} = 0.909593 L + 0.132956 K$	0.99748	4.17
TAQ	$\tilde{P} = -0.17404 L + 0.699103 K + 0.010544 L^2 + 0.000706 K^2 - 0.00697 L K$	0.99835	5.55

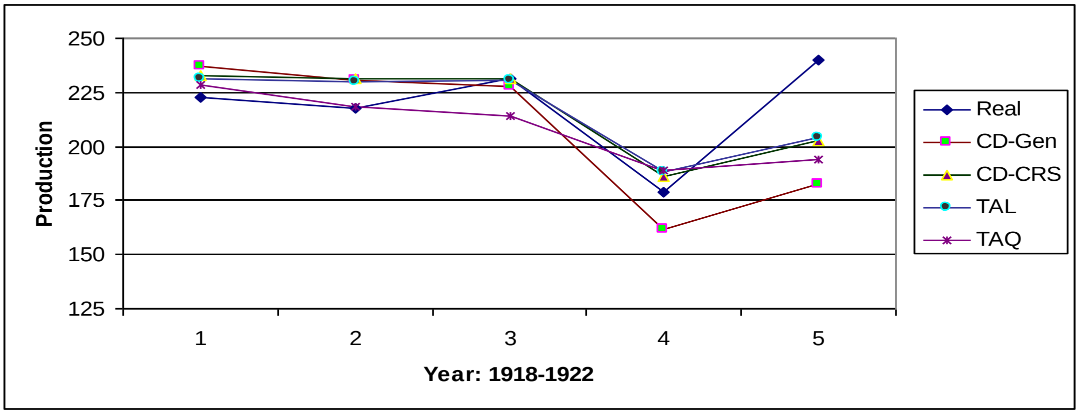

Fig. 4: Combined graphical representation of the estimated models outputs together with the real observations.

Table 8: Some outputs of the estimated models- shown in the table-07, for some real observed input factors.

Year	Real: $\tilde{P}$	L	K
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

CD-Gen	CD-CRS	TAL	TAQ
237.14	232.75	231.49	228.7
230.31	230.98	229.73	218.55
227.69	231.65	230.57	214.22
161.91	186.41	188.24	189.29
182.2	202.7	203.75	194.1

According to the estimated models, shown in table-07 the elasticity parameters of CD-Gen estimated for the whole population and for sample show a remarkable change in values. Importantly only three observations (in 1920-1922) were omitted from the population which left α changed to 1.201113 from 0.768866 and β changed to 0.017984 from 0.247111. Generally it is expected that the changes will be very slow. Comparing the r^2 values, clearly the Taylor approximations are well fitted than the CDs within the sample. Outside the sample the calculated prediction error of the TAL is lower than any other model. With respect to this it stands out that here the TAL is the best fitted model for future prediction using data given in table-05.

In the table-08, the values in the shade boxes are predicted values outside the sample.

The models estimated within the sample (1899-1919) are given in table-07.

According to the outputs outside the sample (1920-1922), the TAL is more efficient than that of CD-Gen.

In figure-04 we provide the combined graphical representation of the estimated models outputs together with the real observations. In the graphs, the last three points represent the efficiency of the estimated models outside the sample and their comparison with the real value in terms of production with respect to the given input factors. The figure-04 shows that none of the estimated model is a good follower of the growth and decay characteristics of the real production. But the overall observation is that the Taylor approximations, particularly the TAL is better than that of CD-Gen.

In this section another real world data has been taken to establish our assumption. The source of data is *the Greek Industrial Sector 1961-1987*. From the prior experiments on real world, we have found some idea on output prediction that our models may not fit too bad within the population. So to avoid the repetition only the prediction efficiency to outside the sample has been shown.

Table 9: Predicted values from 1961 to 1987 from the observed values 1985-1987 for Greek Industrial Sector data.

Year	Real: *P*	*L*	*K*	CD-Gen	CD-CRS	TAL	TAQ
1982	130.388	1016.1	301.9	147.43	146.14	141.21	134.10
1983	130.615	1008.1	314.9	147.13	150.47	145.16	131.92
1984	132.244	985.1	327.7	142.72	154.06	148.42	127.59
1985	137.318	977.1	339.4	142.08	157.79	151.93	125.43
1986	137.468	1007.2	349.492	151.49	162.53	156.49	128.07
1987	135.75	1000	358.231	150.61	165.22	159.07	125.93

Table 10: Estimated results by using Numerical models over the period 1961 to 1984.

	Model Name			
	CD-Gen	CD-CRS	TAL	TAQ
r^2	0.97017	0.88802	0.99128	0.94085
Prediction Error.	12.11	25.27	19.32	10.43

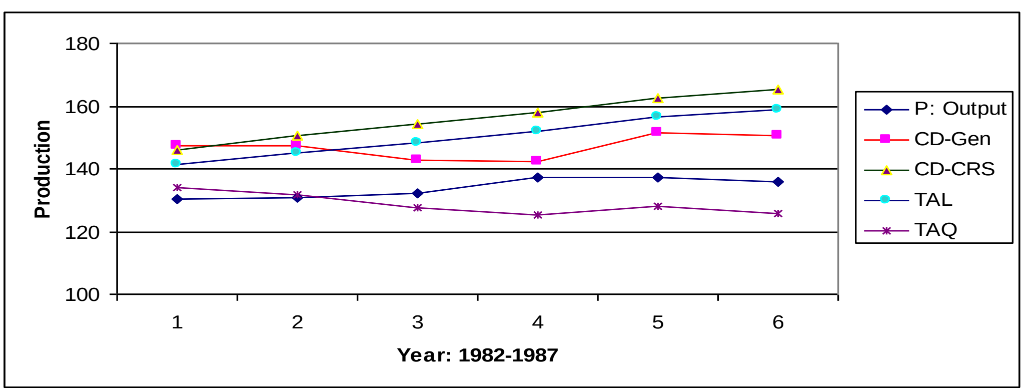

Fig. 5: Graphical representation of observed values together with estimated results for above data.

Similar to the previous tables, in the shaded boxes of the table-09, the predicted outputs of the estimated models outside the sample 1985-1987 are given. Observing the numerical outputs of the estimated models given in table-09 and the statistical measurement scale- the r^2 value, we can say within the sample (1961-1984) the Taylor series approximations are better than the CDs.

It is observed that in full length graphical representation of the models, the TAL model is better fitted within the sample. Though according to the calculated values of prediction errors, outside the sample (shown in table-10), the CD-Gen and TAQ models are well fitted but the prediction error of TAL is not so large compare to others to say that it gives a bad fit. It is also a good enough model according to its growth nature, predicted outputs (shown in figure-05 and table-10) and fitting efficiency within the sample.

6. Discussion and Comparative Nature of the Models

6.1 On Cobb-Douglas Models: The generalized and the CRS Cobb-Douglas models are basically exponential models. The growth and decay of the modes will be rapid (exponential) if the parameters ($\alpha \& \beta$) are not comparatively small and their sum close to 1. Besides, according to our natural perceptions- production, labor and capital are correlated. So it is matter of time to have growth or decay. Hence, the rapid growth or decay cannot be supported naturally. No definite methodology is established with the theory of CD functions to determine the parameter values and their limit. So, the general theories of regression as (4.4) are used to estimate them. It was clear from the numerical outputs, given in the tables that the values of the parameters may change significantly for the same series of observations. This variation occurs due to sample size to estimate the CD models. But it is unexpected to see for a same series of observations such as for a single company or country. The empirical output made contradiction with the assumption of unitary elasticity. It has inflexibility in functional form, such as it should be log-linearized in parameters though the real models are non-linear in variables.

6.2 On Taylor Approximations: The Taylor approximation in lower order such as linear approximation is naturally slow in growth if its values of the parameters are not sufficiently large. That is naturally expected. It is more manageable and flexible in functional form and is handy in empirical estimation. The higher order Taylor series approximations contain the TAL as their initial terms. We analyzed empirically that the TAL fits better though it is a part of all higher orders. Also the higher order part of the Taylor approximations are not needed significantly to make-up the lack of data fitting in the TAL. Another limitation in higher order approximation is that the same input factors used several times in different order. Those repetitions in use of the input factors make the prediction function inflexible. So it is convenient to use the TAL as the representative of Taylor approximations.

Conclusion

The numerical models are estimated using available technology of regression. The aim is to have a comparative analysis between them. Here, the most concentration was on the TAL rather than other higher orders Taylor approximations. Here second ordered Taylor approximation is taken to give evidence about the efficiency of the higher order Taylor approximation. We have seen that the higher order approximations are convergent in terms of output generation and statistical measurement. Having the slow growth nature and flexibility in functional form, the TAL may be the convenient model for production prediction. These are clear in our model's predicted outputs which are shown in the tables and graphical representations. The CD functions are well enough to ensure the data fitting. But they are not rigid enough to maintain the model assumptions with respect to parameters determination; through the available regression theory. It is not our conclusion that the TAL is the best model as a production predictor in terms of the input factors than that of the CDs. The purpose of this article is to establish whether the Cobb-Douglas is the only model to ensure the relationship between the input-output or not. And here our finding is that the Taylor approximations or even the Taylor approximation to linear (TAL) can be a competitive alternative of the CD model.

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