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Exact solution of the relativistic Schroedinger equation for the central complex potential V(r)=i(ar+b/r)

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Abstract

A set of exact solutions of the relativistic Schroedinger equation for the central complex potential V(r)=i(a r+b/r), where *a* and *b* are parameters of the given potential is obtained, by using a suitable ansatz. The exact solution of such equation is found to be relevant in particle physics as well as nuclear physics. The energy eigenvalue and corresponding eigenfunction are obtained for each solution. These solutions are valid, in general when the interrelation between the parameters of the potential (*a* and *b*) and the orbital angular momentum quantum number *l* must be satisfied. These solutions, besides having an aesthetic appeal can be used as bench mark to test the accuracy and reliability of non perturbative methods, which sometimes yield wrong results, of solving the Schroedinger equation.

Keywords: Schroedinger equation, Exact solution, Complex potential

Introduction

One of the important issues of quantum mechanics is to solve the Schroedinger equation for central potentials of physical interest. Unfortunately, however, only for little physical potential, i.e. Coulomb, harmonic oscillator etc., Schroedinger equation could be exactly solved. Considerable efforts have been made in recent years towards obtaining the exact solution of the Schroedinger equation for potentials of physical interest[1-3,5-7,9,17,18]. It has recently been shown that the Schroedinger equation for different class of physical potentials can be solved by choosing a proper ansatz for the eigenfunction.

It is well known that the exact solution of the fundamental dynamical equations play a crucial role in physics. As we know, the exact solution of the Schroedinger equation is possible only for a few potentials and some approximation methods which are frequently used to obtain the solution. Neverthless, it is possible to obtain the exact solution of the Schroedinger equation with the central potentials by applying an ansatz (a method to solve such problems) to the eigenfunction and admitting restrictions on the parameters of the potentials[2,11,16,17,18]. In the past several decades many efforts have been generated in the literature to study the stationary Schroedinger equation with the central potentials containing negative powers of the radial coordinate[1,10,12-15]. As yet, the study of higher order central potentials has been much more desirable to physicists and mathematicians. The interest in these anharmonic-like interactions stems from fact that the study of the relevant Schroedinger equation provides us with insight into the physical problem in question.

In the present work, the exact bound state solutions of the relativistic Schroedinger equation have been obtained using suitable ansatz for the central complex potential V(r)=i(ar+b/r),where the first term is linear in r and the second term is Coulomb like.Complex potentials have been relevant in nuclear and particle physics.The linear and Coulomb potential has been found to be relevant in quarkonium physics[4].For explaining the states of light quarks,one should use the solution of relativistic rather than non relativistic Schroedinger euation.This has been an important issue in quarkonium physics in last three decades.

Schroedinger Equation and Exact Solutions

For exact solution of the relativistic Schroedinger equation, the central complex potential is given as

$$V(r) = i(ar + b/r) \quad \ldots(1)$$

where a and b are parameters of the potential and $i^2=-1$

Using a suitable ansatz, a set of infinite number of exact solutions of the relativistic Schroedinger equation for the complex potential (1) has been obtained.

Consider a relativistic Schrodinger equation (for convenience, set $\hbar$ =1=c):

$$(-\nabla^2 + m^2)\Psi = [E - V(r)]^2\Psi \quad \ldots(2)$$

where ∇^2= Laplacian operator in spherical polar co-ordinate (r,θ,ϕ)

$$\nabla^2 = \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

where $V(r)$ = potential energy; m= mass of the particle; $\hbar = h/2\pi$; and h= Planck's constant.

The reduced radial part of the Schroedinger equation for the potential *V(r) = i(ar + b/r)* is given by:

$$\frac{d^2R_l(r)}{dr^2} + [\{E - V(r)\}^2 - m^2 - \frac{l(l+1)}{r^2}]R_l(r) = 0 \quad \ldots(3)$$

where $R(r)$= radial part of wave function ψ (r, θ, ϕ) and l=orbital angular momentum quantum number.

To solve second order differential Eq. (3), the ansatz used is:

$$R(r) = \exp\left[\frac{1}{2}\alpha r^2 + \beta r\right]\sum_{n=0}^{\infty} a_n r^{n+\nu} \quad \ldots(4)$$

where α, β and ν are constants and are to be determined.

Using Eq. (4) in Eq. (3) and on equating the coefficient of $r^{n+\nu}$ on both side of Eq.(3), the three terms recurrence relation obtained is as follows:

$$A_n a_n + B_{n+1} a_{n+1} + C_{n+2}\, a_{n+2} = 0 \quad \ldots(5)$$

Where,

$$A_n = \beta^2 + (2n + 2\nu + 1)\alpha + E^2 + (2ab) - m^2 \quad \ldots(6a)$$

$$B_n = (2n+2\nu)\beta - i(2bE) \quad \ldots(6b)$$

and

$$C_n = (n+\nu)(n+\nu-1) + b^2 - l(l+1) \quad \ldots(6c)$$

Take $\alpha^2 - a^2 = 0 \quad \ldots(7)$

and $\alpha\beta - i\,Ea = 0 \quad \ldots(8)$

The acceptable values of α and β can be obtained by using Eqs. (7) and (8) and are as follows:

$$\alpha^2 - a^2 = 0 \Rightarrow \alpha^2 = a^2 \Rightarrow \alpha = \pm\sqrt{a}$$

For satisfactory solution, $a \geq 0$; take $\alpha = -\sqrt{a}$ …(9)

$\beta = (iEa/\alpha) = -iE\sqrt{a}$, for $a \geq 0$ …(10)

so that $\left[\frac{R(r)}{r}\right]$ is finite for $r \to \infty$.

Now, if a_0 is the first non vanishing coefficient in the Eq. (4), then by using Eqs. (5) and (6c), one gets:

$C_0=0 \quad \Rightarrow \quad \nu(\nu-1) + b^2 - l(l+1) = 0$

$\Rightarrow \quad \nu^2-\nu - [l(l+1)-b^2]=0$

which is quadratic in ν and hence

$$\nu = \nu_{\pm} = \frac{1}{2} \pm \sqrt{(l+\frac{1}{2})^2 - b^2} \quad \ldots(11)$$

Where ν=physically acceptable

Again, if a_p is the last non vanishing coefficient in the Eq. (4), i.e. $a_p \neq 0$, but $a_{p+1}=0=a_{p+2}=$ ………………………., then the recurrence Eq. (5) gives:

$$A_p\, a_p = 0 \text{ since } a_p \neq 0 \Rightarrow A_p = 0 \quad \ldots(12)$$

Eq. (6a) gives:

$$\beta^2 + (2\nu + 2p + 1)\,\alpha + E_p^2 + 2ab - m^2 = 0 \quad \ldots(13)$$

Putting $\alpha = -\sqrt{a}$ *and* $\beta = -iE_p\sqrt{a}$ *and solving for* E_p *gives:*

$$E_p = \pm[\{2ab - m^2 - (2p + 2\nu_{\pm} + 1)\sqrt{a}\}(a-1)]^{1/2} \quad \ldots(14)$$

where the + sign has to be chosen so that we get a positive value for the rest mass for free particle.

Further, non trivial solution of coefficients a_n's from the recurrence relation given by Eq. (5) must satisfy the determinant relation given by:

$$\begin{vmatrix} B_0 & C_1 & 0 & 0 & 0 \\ A_0 & B_1 & C_2 & 0 & 0 \\ 0 & 0 & A_{p-2} & B_{p-1} & C_p \\ 0 & 0 & 0 & A_{p-1} & B_p \end{vmatrix} = 0 \quad \ldots(15)$$

The different exact solutions can be generated by setting $p = 0, 1, 2,$ ….etc. Thus there are mainly two following cases:

*CaseI***:** For $p = 0$, from Eq. (14), the energy eigenvalue of the problem is given by:

$$E_0 = [\{2ab - m^2 - (2\nu_{\pm} + 1)\sqrt{a}\}/(a-1)]^{1/2} \quad \ldots(16)$$

and the complete eigenfunction can be obtained by using Eqs. (9) - (11) and (16) in Eq. (2) as :

$$\Psi_{o,l,m}(r,\theta,\phi) = \exp\left[-\frac{1}{2}r^2\sqrt{a} - E_0 r\sqrt{a}\right] a_0 r^{(\nu_{\pm}-1)} Y_{l,m}(\theta,\phi) \quad \ldots(17)$$

The Eq. (15) together with the Eqs. (9) - (11) and (16) imply the interrelation given by:

$$-\left[\frac{1}{2} \pm \left\{\left(l+\frac{1}{2}\right)^2 - b^2\right\}^{\frac{1}{2}}\right]\sqrt{a} + 2ab + m^2 = 0 \quad \ldots(18)$$

between the parameters a and b of the potential and the orbital angular momentum quantum number l.

*Case II***:** For $p = 1$, in Eq. (14), the energy eigen value is:

$$E_1 = \pm[\{2ab - m^2 - (2\nu_{\pm} + 3)\sqrt{a}\}/(a-1)]^{1/2} \quad \ldots(19)$$

The complete eigenfunction can be obtained by using Eqs. (9) - (11) and (19) and is given by:

$$\Psi_{1,l,m}(r,\theta,\phi) = \left[\exp\left\{-\frac{1}{2}r^2\sqrt{a} - iE_1 r\sqrt{a}\right\}\right].\left[a_0 r^{(\nu_{\pm}-1)} + a_1 r^{\nu_{\pm}}\right].Y_{l,m}(\theta,\phi) \quad \ldots(20)$$

By using Eq. (5), the expansion coefficients a_0 and a_1 are related by the relation

$$(\sqrt{a})a_0 - i\{(\nu_+ + 1)\sqrt{a} + b\}E_1 a_1 = 0 \quad \ldots(21)$$

In this case, determinant of Eq. (15) leads to

$$\begin{vmatrix} B_0 & C_1 \\ A_0 & B_1 \end{vmatrix} = 0 \quad \Rightarrow \quad B_0 B_1 - A_0 C_1 = 0 \quad \ldots(22)$$

Eq. (22) gives

$$A_0 = \beta^2 + (2\nu+1)\alpha + (E_0)^2 + 2ab - m^2 \quad \text{...(23a)}$$

$$B_o = 2\nu\beta - i(2bE_0) \quad \text{...(23b)}$$

$$B_1 = (2\nu+2)\beta - i(2bE_1) \quad \text{...(23c)}$$

$$C_1 = \nu(\nu-1) + b^2 - l(l+1) \quad \text{...(23d)}$$

On putting $\alpha = -\sqrt{a}$ and $\beta = -E\sqrt{a}$; and by using Eq. (23) in Eq. (22) and simplifying, one gets

$$-(3+2\nu_+)\sqrt{a} + 2ab - m^2 = 0 \quad \text{...(24)}$$

Eq. (24) is the interrelationship between parameters *a* and *b* of the potential and the orbital angular momentum quantum number *l*.

Results and Discussion

The explicit expressions of energy eigenvalue and eigenfunction are obtained for each solution. These solutions are valid when for, in general, each solution has an interrelation between the parameters of the potential and the orbital angular momentum quantum number (*l*) is satisfied. These solutions, besides having an aesthetic appeal, can be used as bench mark to test the accuracy and reliability of nonperturbative methods, which sometimes yield wrong result, of solving the Schroedinger equation. The central complex potential considered in this problem may be relevant in nuclear and particle physics.

The central irregular singular potentials $V(r)$ for which $\lim_{r\to 0} r^2 V(r) \to \infty$ are mathematically more difficult to treat. With development of fast computers, it is although possible to solve the Schroedinger equation, numerically, for any potential, to a desired degree of accuracy, the exact solutions however have an aesthetic appeal. Further the exact solutions can serve as bench mark to test the accuracy and reliability of various nonperturbative methods of solving the Schroedinger equation.

In the present study, a set, consisting of eigenvalues and the corresponding eigenfunctions, of exact solutions of the Schroedinger equation for the central complex potential *V(r)=i(ar + b/r)*, where the first term is linear in *r* and the second term in Coulomb like. Complex potentials have been relevant in nuclear and particle physics. The linear and Coulomb potential has been found to be relevant in quarkonium physics[4] . For explaining the states of light quarks, one should use the solutions of relativistic rather than non relativistic Schroedinger equation. This has been an important issue in quarkonium physics in last three decades.

Conclusions

The exact bound state solutions of Schroedinger equation (relativistic) for the central complex potential *V(r)=i(ar + b/r)* by using a suitable ansatz have been obtained. The solutions obtained hold when for each solution, a separate relation interrelating the parameters *a* and *b* of potential and orbital angular momentum quantum number (*l*) is satisfied. The eigenfunctions, which are obtained in closed form, are also square integrable. Such solution of relativistic Schroedinger equation plays most important role in nuclear and particle physics and especially in quarkonium physics.

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References

1. Bose S K, *J Math Phys Sci*, 26 (1992) 129.
2. Bose S K & Gupta N, *IL Nuovo Cimento*, 113 B, No 3 (1998) 299.
3. Schulze– Halberg A, *Hadronic J*, 22 (1999).
4. Martin A, *Phys Lett*, 100 B (1981) 511.
5. Bose S K, *Phys Lett*, 141 (1989) 141.
6. Bose S K, *Phys Lett*, 147 (1990) 85.
7. Bose S K, *Fizika*, 23 (1991) 63.
8. Frank W M, Land D J & Spector R M, *Rev Mod Phys*, 43 (1971) 36.
9. De Souza Dutra A & Boschi Filho H, *Phys Rev A*, 44 (1991) 4721.
10. Khare A & Behra S N, *Pramana J Phys*, 4 (1980) 327.
11. Dong Shi-Hai & Ma Zhong-Qi, *J Phys A*, 31 (1998) 9855.
12. Dong Shi-Hai, Ma Zhong-Qi & Esposito G, *Found Phys Lett*, 12 (1999) 465.
13. Esposito G, *J Phys A*, 31 (1998) 9493.
14. Vogt E & Wannier G H, *Phys Rev*, 95 (1954) 1190.
15. Voros A, *J Phys A*, 32 (1999) 5993.
16. Bose S K, *IL Nuovo Cimento*, 112 B, No 4 (1997) 635.