

A Schematic Transformation for the Non relativistic Schroedinger Equation

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Abstract

A useful transformation which interrelates a large class of non relativistic Schroedinger equation is to be obtained. It is possible to obtain the ground state of non relativistic Schroedinger equations. The exact solutions may be used to act the accuracy and the reliability of numerical methods of solving Schroedinger equation. In these transformations, for a class of potentials, Schroedinger equations are interrelated. Such an interrelation is useful for finding exact solution of these Schroedinger equations.

Keywords: Schroedinger equation, Ground state, Exact solution, Transformation

Introduction

One of the important issues of quantum mechanics is to solve the Schroedinger equation for physical potentials. Unfortunately, however, only for few physical potentials e.g. Coulomb, harmonic oscillator etc., Schroedinger equation could be exactly solved. Considerable effort has been made in recent years towards obtaining the exact solution of the Schroedinger equation for potentials of physical interest. It has recently been shown that the Schroedinger equation for a very large class of physical potentials can be

solved by choosing a proper ansatz for the eigen function [1-3, 5, 6].

In the present work, using a simple transformation, we show that for a class of potentials, the Schroedinger equations are interrelated. Such an interrelation is useful for finding exact solution of these Schroedinger equations [4, 7, 8].

Schroedinger equation and exact solutions

For exact solution of nonrelativistic Schroedinger equation for various potential of physical interest, consider the Schroedinger equation and is given by

$$\nabla^2 \Psi + \frac{2m}{\hbar^2} [E - V(r)] \Psi = 0 \dots\dots\dots (1)$$

Where $\nabla^2 =$ Laplacian operator

$$\nabla^2 = \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \Phi^2} \right] \dots\dots (2)$$

$\Psi = \text{Normalised wave function}$

$m = \text{mass of particle}$

$E = \text{total energy}$

$V(r) = \text{Potential energy}$

$$\hbar = \frac{h}{2\pi}$$

and $h = \text{Planck's constant}$

Now consider the reduced radial part of Schroedinger equation and is given by

$$\left[-\frac{\hbar^2}{2m} \left\{ \frac{d^2}{dr_o^2} - \frac{1}{4r_o^2} (L_o - 1)(L_o - 3) \right\} + \{V(r_o) - E\} \right] u_o(r_o) = 0 \dots (3)$$

in N-dimension with central potential $V(r_o)$ where $L_o = (N + 2l)$.

If we use the transformation

$$r_o = (r_1)^{m_o} \text{ and } u_o(r_o) = r_1^{n_o} u_1(r_1) \dots (4)$$

then the equation (3) becomes

$$\left[-\frac{\hbar^2}{2m} \left\{ \frac{d^2}{dr_1^2} - \frac{1}{4r_1^2} (L_1 - 1)(L_1 - 3) \right\} + (2n_o + 1)^2 r_1^{2\{(2n_o+1)-1\}} \{V(r_1) - E\} \right] u_1(r_1) = 0 \dots (5)$$

Where we have set $m_o = (2n_o + 1)$ and $L_1 = \{(2n_o + 1)(L_o - 1) - (2n_o - 1)\} \dots (6)$

The equation (5) is again in the form of Schroedinger equation for the central potential

$$\left[(2n_o + 1)^2 r_1^{2\{(2n_o+1)-1\}} \{V(r_1) - E\} \right] \text{ for zero energy eigenvalue.}$$

Using this transformation once again i.e.

$$\text{by setting } r_1 = (r_2)^{m_1} \text{ and } u_1(r_1) = r_2^{n_1} u_2(r_2) \dots (7) \text{ in}$$

the equation (5), we get

$$\left[-\frac{\hbar^2}{2m} \left\{ \frac{d^2}{dr_2^2} - \frac{1}{4r_2^2} (L_2 - 1)(L_2 - 3) \right\} + \{(2n_o + 1)(2n_1 + 1)\}^2 r_2^{2\{(2n_o+1)(2n_1+1)-1\}} \{V(r_2) - E\} \right] u_2(r_2) = 0 \dots (8)$$

Where we have set

$$m_2 = (2n_2 + 1) \text{ and } L_3 = [(2n_2 + 1)(L_2 - 1) - (2n_2 - 1)] \dots (9)$$

It is observed that, equation (8) is once again in the form of a Schroedinger equation for the central potential

$$\left[(2n_o + 1)(2n_1 + 1)\}^2 r_2^{2\{(2n_o+1)(2n_1+1)-1\}} \{V(r_2) - E\} \right] \text{ with energy eigenvalue}$$

zero. By using this transformation, equation (8), by setting

$$r_2 = (r_3)^{m_2} \text{ and } u_2(r_2) = r_3^{n_2} u_3(r_3) \dots (10)$$

We have the equation

$$\left[-\frac{\hbar^2}{2m} \left\{ \frac{d^2}{dr_3^2} - \frac{1}{4r_3^2} (L_3 - 1)(L_3 - 3) \right\} + \{(2n_o + 1)(2n_1 + 1)(2n_2 + 1)\}^2 r_3^{2\{(2n_o+1)(2n_1+1)(2n_2+1)-1\}} \{V(r_3) - E\} \right] u_3(r_3) = 0 \dots (11)$$

The equation (11) is again in the form of a Schrodinger equation with zero energy eigenvalue for the central potential

$$\left[\{(2n_o + 1)(2n_1 + 1)(2n_2 + 1)\}^2 r_3^{2\{(2n_o+1)(2n_1+1)(2n_2+1)-1\}} \{V(r_3) - E\} \right]$$

In this way, in general, equation (11) can be rewritten as

$$\left[-\frac{\hbar^2}{2m} \left\{ \frac{d^2}{dr_p^2} - \frac{1}{4r_p^2} (L_p - 1)(L_p - 3) \right\} + \prod_{i=0}^{p-1} (2n_i + 1)^2 (r_p)^{2\{\prod_{i=0}^{p-1} (2n_i+1)-1\}} \{V(r_p) - E\} \right] u_p(r_p) = 0 \dots (12)$$

Here p=0, 1, 2, 3,

with zero energy eigenvalue for the central potential

$$\prod_{i=0}^{p-1} (2n_i + 1)^2 (r_p)^{2\{\prod_{i=0}^{p-1} (2n_i+1)-1\}} [V(r_p) - E].$$

The above transformations are quite general and do not depend on the form of $V(r_o), u_o(r_o)$ and E .

For a practical application of the present method, we first consider the Schrodinger equation for a potential, whose solutions can be exactly found. A well familiar example, is the case of Coulomb potential,

$$V(r_o) = -\frac{\alpha}{r_o}, \dots (13)$$

where $\alpha = \text{constant}$, for which both the exact energy eigenvalue

$$E_n = -\frac{\alpha^2}{2n^2}$$

(set $m = c = \hbar = 1$, for convenience) and the eigenfunction is

$$u_o(r_o) = C_o r_o^{\frac{L_o-1}{2}} . e^{-\frac{\alpha r_o}{n}} . F\left(-n + \frac{L_o - 1}{2}, L_o - 1, \frac{2\alpha r_o}{n}\right) \dots (14)$$

Here $c_0 = \text{constant}$ are known

In the present case, the ground state eigenfunction is

$$\{u_o(r_o)\}_{ground\ state} = C_o r_o^n . e^{-\frac{\alpha r_o}{n}} \dots (15)$$

The Schrodinger equation for the exactly solvable Coulomb potential can be written as

$$\left[-\frac{1}{2} \left\{ \frac{d^2}{dr_o^2} - \frac{1}{4r_o^2} (L_o - 1)(L_o - 3) \right\} - \frac{\alpha}{r_o} + \frac{\alpha^2}{2n^2} \right] u_o(r_o) = 0 \dots (16)$$

In view of equation (16), the equation (5) can be written as

$$\left[-\frac{1}{2} \left\{ \frac{d^2}{dr_1^2} - \frac{1}{4r_1^2} (L_1 - 1)(L_1 - 3) \right\} + (2n_o + 1)^2 \left\{ -\alpha r_1^{(2n_o-1)} + \frac{\alpha^2}{2n^2} r_1^{4n_o} \right\} \right] u_1(r_1) = 0 \dots (17)$$

By using a little bit of algebraic manipulation, the ground state eigenfunction for the Schroedinger equation (17) is given by

$$u_1(r_1) = c_1 r_1^{q_0} \exp\left(-\frac{\alpha}{n} r_1^{2n_0+1}\right) \dots\dots\dots (18)$$

$$\text{where } q_0 = \left[\frac{1}{2}(2n_0 + 1)(2n - 1) + \frac{1}{2}\right] \dots\dots\dots (19)$$

In this case, equation (8) leads to

$$\left[-\frac{1}{2}\left\{\frac{d^2}{dr_2^2} - \frac{1}{4r_2^2}(L_2 - 1)(L_2 - 3)\right\} + \{(2n_0 + 1)(2n_1 + 1)^2\}\left\{-\alpha r_2^{2N_1-1} + \frac{\alpha^2}{2n^2} r_2^{4N_1}\right\}\right] u_2(r_2) = 0 \dots (20)$$

$$\text{Where } 2N_1 + 1 = (2n_0 + 1)(2n_1 + 1) \dots\dots\dots (21)$$

By using a little bit of algebraic manipulation, the ground state eigenfunction for Schroedinger equation (20) is given by

$$u_2(r_2) = c_2 r_2^{q_1} \cdot \exp\left(-\frac{\alpha}{n} r_2^{2N_1+1}\right) \dots\dots\dots (22)$$

$$\text{where } q_1 = \left[\frac{1}{2}(2N_1 + 1)(2n - 1) + \frac{1}{2}\right] \dots\dots\dots (23)$$

In similar way, equation (11) leads to

$$\left[-\frac{1}{2}\left\{\frac{d^2}{dr_3^2} - \frac{1}{4r_3^2}(L_3 - 1)(L_3 - 3)\right\} + \{(2n_0 + 1)(2n_1 + 1)(2n_2 + 1)\}^2\left\{-\alpha r_3^{2N_2-1} + \frac{\alpha^2}{2n^2} r_3^{4N_2}\right\}\right] u_3(r_3) = 0 \dots (24)$$

$$\text{Where } 2N_2 + 1 = (2n_0 + 1)(2n_1 + 1)(2n_2 + 1) \dots\dots\dots (25)$$

We obtain the exact ground state eigenfunction for the Schroedinger equation (24), after doing the similar algebraic manipulations, as

$$u_3(r_3) = c_3 r_3^{q_2} \cdot \exp\left(-\frac{\alpha}{n} r_3^{2N_2+1}\right) \dots\dots\dots (26)$$

$$\text{where } q_2 = \left[\frac{1}{2}(2N_2 + 1)(2n - 1) + \frac{1}{2}\right] \dots\dots\dots (27)$$

In the last, Schroedinger equation (12) leads to

$$\left[-\frac{1}{2}\left\{\frac{d^2}{dr_p^2} - \frac{1}{4r_p^2}(L_p - 1)(L_p - 3)\right\} + \{2N_{p-1} + 1\}^2\left\{-\alpha r_p^{2N_{p-1}+1} + \frac{\alpha^2}{2n^2} r_p^{4N_{p-1}}\right\}\right] u_p(r_p) = 0 \dots (28)$$

$$\text{Where } \{2N_{p-1} + 1\} = \prod_{i=0}^{p-1} (2n_i + 1) \dots\dots\dots (29)$$

Following the above method, the ground state eigenfunctions for Schroedinger equation (28) can be written as

$$u_p(r_p) = C_p r_p^{q_{p-1}} \cdot \exp\left(-\frac{\alpha}{n} r_p^{2N_{p-1}+1}\right) \dots\dots\dots (30)$$

$$\text{where } q_{p-1} = \left[\frac{1}{2}\{2N_{p-1} + 1\}(2n - 1) + \frac{1}{2}\right] \dots\dots\dots (31)$$

Results and Discussion

The exact eigenfunction for different Schroedinger equation for zero energy eigenvalue obtained above can be used to test the accuracy of numerical methods of solving Schroedinger equation. The Schroedinger equation (16) for the Coulomb potential possesses an infinite number of exact solutions and these may be used to find exact solutions of the related equations (5), (8), (11) and (12) by following the above methods.

By using such method, we can find the exact eigenfunction for Schroedinger equation for the central fraction power potentials for zero energy eigenvalue.

To illustrate it, let us put $n_o = \frac{1}{4}$ in equations (17), then Schroedinger equation becomes

$$\left[-\frac{1}{2} \left\{ \frac{d^2}{dr_1^2} - \frac{1}{4r_1^2} (L_1 - 1)(L_1 - 3) \right\} + \left(\frac{3}{2} \right)^2 \left\{ -\alpha r_1^{-\frac{1}{2}} + \frac{\alpha^2}{2n^2} r_1 \right\} \right] u_1(r_1) = 0 \dots (32)$$

For the fractional power potential =

$$V(r_1) = \left(\frac{3}{2} \right)^2 \left\{ -\alpha r_1^{-\frac{1}{2}} + \frac{\alpha^2}{2n^2} r_1 \right\} \dots \dots (33)$$

for the zero energy eigenvalue.

Using above method, exact eigenfunction for the Schroedinger equation (32) can be obtained as

$$u_1(r_1) = C_1 r_1^{\left(\frac{3}{2}n - \frac{1}{4}\right)} \cdot \exp\left(-\frac{\alpha}{n} r_1^{\frac{3}{2}}\right) \dots \dots \dots (34)$$

Here $C_1 = \text{constant}$

Taking different fractional values of $n_o, n_1, n_2 \dots \dots n_i, \dots \dots etc.$, the exact ground state wave functions of the Schroedinger equations for a very large class of central fractional power potentials can be obtained.

Conclusion

In this research paper, we obtain exact eigenfunctions for Schroedinger equation using an interesting and useful transformation for various central potentials of physical interest for zero energy eigenvalue. We obtained the ground state solution of non relativistic Schroedinger equation and may be used to check the accuracy and the reliability of numerical methods of solving Schroedinger equation. On taking different fractional values of $n_o, n_1, n_2 \dots \dots n_i, \dots \dots etc.$, the exact ground state wave function of the Schroedinger equation for a large class of central fractional power potentials can be obtained. In future, such transformation can be used to obtain the ground state eigenfunction for the relativistic Schroedinger equation for

various central potentials of physical importance.

Acknowledgement

The author gratefully acknowledges Prof H.C. Prasad, Department of Physics, DDU Gorakhpur University, Gorakhpur for encouragement and giving necessary suggestions.

References

1. Srivastava V K and Bose S K, Indian J. Pure & Applied Phys.,47(2009) 547
2. Srivastava V K and Bose S K, Int. J. Sci. & App. Research, 4(2017)99
3. Srivastava V K and Bose S K, G S R J M , 3 (2017) 16

4. Bose S K and Varma N, Phys. Lett., 141 (1989) 141. Phys. Lett., 147 (1990) 85. Fizika , 23 (1991) 63
5. Bose S K and Halberg A S, Physica Scripta, 29 (2000) 81
6. Dutra A D S and Filho H B, Phys. Rev., 44A(1991) 4721
7. Bose S K, J. Math and Phys. Sci., 26 (1992) 129. Nuov. Cim., 109 (1994) 1217. J. Phys. Mal., 14 (1993) 125
8. Bose S K and Gupta N, Nuov, Cim., 113 B (1998) 299.