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Generalized Differential Transform Method for non-linear Inhomogeneous Time Fractional Partial differential Equation

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Abstract

In the present paper, we obtain the approximate solutions of non-linear inhomogeneous time-fractional partial differential equation by Generalized Differential Transform Method (GDTM). The fractional derivatives are described in the Caputosense. GDTM provides an analytical solution in the form of an infinite power series with easily computable components. GDTM can solve the linear and nonlinear fractional differential equations with negligible error compared to the exact solution. The method introduces a promising tool for solving many linear and nonlinear fractional differential equations.

Keywords: Fractional differential equations; Caputo fractional derivative; Generalized Differential transform method; Generalized Taylor formula; Analytic solution

Mathematical Subject Classification (2010): 26A33, 34A08, 35A22, 35K25, 35R11, 74H10.

Introduction

In the present analysis GDTM have been used tosolve the followingnon-linear inhomogeneous time-fractional partial differential equation

$$\frac{{}^{c}\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+x\frac{\partial u(x,t)}{\partial x}+\frac{\partial^{2}u(x,t)}{\partial x^{2}}+cu^{2}(x,t)=2t^{\alpha}+2x^{2}+2\ \ ;\ \ t>0$$

subject to initial condition $u(x,0)=x^2$

(1.1)

where $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ is the fractional differential operator(Caputo derivative) of order $\alpha\in(0,1]$ and $c=$ constant

It has been found that the solutions obtained by these methods are significantly identical.

Fractional differential equations are generalizations of classical differential equations of integer order and have recently been proved to be valuable tools in the modeling of many physical phenomena in various fields of science and engineering. A lot of works have been done by using fractional derivatives for a better description of considered material properties. Mathematical modeling based on enhanced rheological models naturally leads to differential equations of fractional order and to the necessity of the formulation of the initial conditions to such equations. Recently, various analytical and numerical methods have been employed to solve linear and nonlinear fractional differential equations. The differential transform method (DTM) was proposed by Zhou [1]to solve linear and nonlinear initial value problems in electric circuit analysis. This method has been used for solving various types of equations by many authors[2-16] .DTM constructs an analytical solution in the form of a polynomial and different from the traditional higher order Taylor series method. For solving two-dimensional linear and nonlinear partial differential equations of fractional order DTM is further developed as Generalized Differential Transform Method (GDTM)by Momani, Odibat, and Erturk in their papers [17-19]. Recently, VedatSuat Ertiirka and Shaher Momanib applied generalized differential transform method to solve fractional integro-differential equations[20].The GDTM is implemented to derive the solution of space-time fractional telegraph equation by MridulaGarg, Pratibha Manohar and Shyam L. Kalla[21].Manish Kumar Bansal, Rashmi Jain applied generalized differential transform method to solve fractional order Riccati differential equation [22].

Mathematical Preliminaries on Fractional Calculus:

To solve the problems of fractional differential equations many definitions of fractional calculus have been developed. The most frequently encountered definitions include Riemann-Liouville, Caputo, Wely, Rize fractional operator. We introduce the following definitions[23, 24] in the present analysis:

1. Definition: Let $\alpha \in \boldsymbol{R}^+$ .The integral operatorI^α defined on the usual Lebesgue space L (a,b) by

$$I^\alpha f(x) = \frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

$$I^0 f(x) = f(x),$$

for x $\in [a,b]$ is called Riemann-Liouville fractional integral operator of order α ($\geq$0).

It has the following properties:

(i) $I^\alpha f(x)$exists for any $x \epsilon [a,b]$

$(ii) I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$

(iii) $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x)$

(iv) $I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$,

where$f(x) \epsilon L[a,b], \alpha, \beta \geq 0, \gamma > -1$

2. Definition: The Riemann-Liouville definition of fractional order derivative is
$$ {}^{RL}_{0}D^{\alpha}_{x} f(x) = \frac{d^n}{dx^n}\ {}_{0}I^{n-\alpha}_{x} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\,dt , $$
where n is an integer that satisfies $\alpha \in (n-1, n)$

3. Definition: A modified fractional differential operator ${}^{c}_{0}D^{\alpha}_{x}$ proposed by Caputo is given by
$$ {}^{c}_{0}D^{\alpha}_{x} f(x) = {}_{0}I^{n-\alpha}_{x} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, $$
where $\alpha \in \boldsymbol{R}^{+}$ is the order of operation and n is an integer that satisfies $\alpha \in (n-1, n)$.
It has the following two basic properties[25]:
(i) If $f \in L_{\infty}(a,b)$ or $f \in C[a,b]$ and $\alpha > 0$ then
$$ {}^{c}_{0}D^{\alpha}_{x}\ {}_{0}I^{\alpha}_{x} f(x) = f(x) $$
(ii) If $f \in C^{n}[a,b]$ and if $\alpha > 0$, then
$$ {}_{0}I^{\alpha}_{x}\, {}^{c}_{0}D^{\alpha}_{x} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} x^k \ ; \alpha \in (n-1, n) $$

4. Definition: For *m* being the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$, is defined as in [26]
$$ D^{\alpha}_{t} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{\partial^m u(x,\xi)}{\partial \xi^m} \ ; & \alpha = m \epsilon \mathbb{N} \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\xi)^{m-\alpha-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m}\, d\xi \ ; & m-1 \le \alpha < m \end{cases} $$

Relation between Caputo derivative and Riemann-Liouvillederivative:
$$ {}^{c}_{0}D^{\alpha}_{t} f(x) = {}^{RL}_{0}D^{\alpha}_{t} f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \ ; \ \alpha \in (m-1, m) $$
.
Integrating by parts, we get the following formulae as given in [27]:
(i) $\int_a^b g(x)\, {}^{c}_{a}D^{\alpha}_{x} f(x) dx = \int_a^b f(x)\, {}^{RL}_{x}D^{\alpha}_{b} g(x) dx + \sum_{j=0}^{n-1} \left[{}^{RL}_{x}D^{\alpha+j-n}_{b} g(x)\, {}^{RL}_{x}D^{n-1-j}_{b} f(x) \right]_a^b$
(ii) For $n = 1, \int_a^b g(x)\, {}^{c}_{a}D^{\alpha}_{x} f(x) dx = \int_a^b f(x)\, {}^{RL}_{x}D^{\alpha}_{b} g(x) dx + \left[{}_{x}\mathrm{I}^{1-\alpha}_{b} \mathrm{g(x).f(x)} \right]_a^b$

Generalized two dimensional differential transform method:-
Consider a function of two variables $u(x,y)$ be aproduct of two single-variable functions, i.e. $u(x,y) = f(x)g(y)$, which isanalytic and differentiated continuously with respect to *x* and *y* in the domain of interest.Then the generalized two-dimensional differential transform of the function $u(x,y)$ is givenby [17-19]
$$ U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \left[\left(D^{\alpha}_{x_0}\right)^k \left(D^{\beta}_{y_0}\right)^h u(x,y) \right]_{(x_0 y_0)} \quad (3.1) $$
where $0 < \alpha, \beta \le 1$; $U_{\alpha,\beta}(k,h) = F_{\alpha}(k) G_{\beta}(h)$ is called the spectrum of $u(x,y)$ and $\left(D^{\alpha}_{x_0}\right)^k = D^{\alpha}_{x_0}, D^{\alpha}_{x_0}, \ldots\ldots, D^{\alpha}_{x_0}$ (k-times).

The inverse generalized differential transform of $U_{\alpha,\beta}(k,h)$ is given by

$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}$ (3.2)

It has the following properties:

(i) if $u(x,y) = v(x,y) \pm w(x,y)$ then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$

(ii) if $(x,y) = av(x,y)$, $a \in \mathbb{R}$ then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$

(iii) if $u(x,y) = v(x,y)w(x,y)$ then $U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)\, W_{\alpha,\beta}(k-r,s)$

(iv) if $u(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$ then $U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$

(v) if $u(x,y) = D_{x_0}^{\alpha}v(x,y), 0 < \alpha \le 1$ then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+1,h)$

(vi) if $u(x,y) = D_{x_0}^{\gamma}v(x,y), 0 < \gamma \le 1$ then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}\left(k+\frac{\gamma}{\alpha},h\right)$

(vii) if $u(x,y) = D_{y_0}^{\gamma}v(x,y), 0 < \gamma \le 1$ then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta h+\gamma+1)}{\Gamma(\beta h+1)}V_{\alpha,\beta}\left(k,h+\frac{\gamma}{\beta}\right)$

where $U_{\alpha,\beta}(k,h), V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ are the differential transformations of the functions $u(x,y), v(x,y)$ and $w(x,y)$ respectively and

$$\delta(k-n) = \begin{cases} 1 & ; \quad k = n \\ 0 & ; \quad k \ne n \end{cases}$$

Applying generalized two-dimensional differential transform (3.1)with $(x_0,y_0) = (0,0)$ on (1.1) we obtain

$$U_{1,\alpha}(k,h+1) = \frac{\Gamma(\alpha h+1)}{\Gamma(\alpha(h+1)+1)}\Big[-\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-1)\delta(h-s)(k-r+1)\,U_{1,\alpha}(k-r+1,s) - c\sum_{r=0}^{k}\sum_{s=0}^{h}U_{1,\alpha}(r,h-s)\,U_{1,\alpha}(k-r,s) - (k+1)(k+2)U_{1,\alpha}(k+2,h) + 2\delta(h-1)\delta(k) + 2\delta(h)\delta(k-2) + 2\delta(h)\delta(k)\Big]$$

(3.3)

and $U_{1,\alpha}(k,0) = \delta(k-2) = \begin{cases} 1 & ; \quad k = 2 \\ 0 & ; \quad k \ne 2 \end{cases}$ (3.4)

Taking $p = h+1$ in (3.3) we get

$$U_{1,\alpha}(k,p) = \frac{\Gamma(\alpha(p-1)+1)}{\Gamma(\alpha p+1)}\Bigg[-\sum_{r=0}^{k}\sum_{s=0}^{p-1}\delta(r-1)\delta(p-s-1)(k-r+1)\,U_{1,\alpha}(k-r+1,s) - c\sum_{r=0}^{k}\sum_{s=0}^{p-1}U_{1,\alpha}(r,p-s-1)\,U_{1,\alpha}(k-r,s) - (k+1)(k+2)U_{1,\alpha}(k+2,p-1) + 2\delta(p-2)\delta(k) + 2\delta(p-1)\delta(k-2) + 2\delta(p-1)\delta(k)\Bigg]$$

i.e.

$$U_{1,\alpha}(k,h) = \frac{\Gamma(\alpha(h-1)+1)}{\Gamma(\alpha h+1)}\Big[-\sum_{r=0}^{k}\sum_{s=0}^{h-1}\delta(r-1)\delta(h-s-1)(k-r+1)\,U_{1,\alpha}(k-r+1,s) - c\sum_{r=0}^{k}\sum_{s=0}^{h-1}U_{1,\alpha}(r,h-s-1)\,U_{1,\alpha}(k-r,s) - (k+1)(k+2)U_{1,\alpha}(k+2,h-1) + 2\delta(h-2)\delta(k) + 2\delta(h-1)\delta(k-2) + 2\delta(h-1)\delta(k)\Big]$$

(3.5)

Now utilizing the recurrence relation (3.5) and the initial condition (3.4),we obtain after a little simplification the following values of $U_{1,\alpha}(k,h)$ for $k=0,1,2,\dots$ and $h=1,2,3\dots$

$U_{1,\alpha}(2,0)=1;\ U_{1,\alpha}(k,0)=0\ for\ k\in\mathbb{W}-\{2\}$;

$U_{1,\alpha}(0,1)=0;\ U_{1,\alpha}(0,2)=\frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)};\ U_{1,\alpha}(0,3)=0;U_{1,\alpha}(0,4)=0;U_{1,\alpha}(0,5)=-\frac{4c(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1)}$;

$U_{1,\alpha}(0,6)=0;U_{1,\alpha}(0,7)=0;U_{1,\alpha}(0,8)=\frac{16c^2(\Gamma(\alpha+1))^3\Gamma(4\alpha+1)\Gamma(7\alpha+1)}{(\Gamma(2\alpha+1))^3\Gamma(5\alpha+1)\Gamma(8\alpha+1)};U_{1,\alpha}(0,9)=0;U_{1,\alpha}(0,10)=0;$

$U_{1,\alpha}(1,h)=0\ for\ h=1,2,3,\dots$;

$U_{1,\alpha}(2,1)=0;U_{1,\alpha}(2,2)=0;U_{1,\alpha}(2,3)=-\frac{4c\Gamma(\alpha+1)}{\Gamma(3\alpha+1)};\ U_{1,\alpha}(2,4)=0;U_{1,\alpha}(2,5)=0;$

$U_{1,\alpha}(2,6)=\frac{8c^2(\Gamma(\alpha+1))^2\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)\Gamma(2\alpha+1)}\left[\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2}{\Gamma(3\alpha+1)}\right];U_{1,\alpha}(2,7)=0;U_{1,\alpha}(2,8)=0;$

$U_{1,\alpha}(2,9)=-\frac{32c^3(\Gamma(\alpha+1))^3\Gamma(8\alpha+1)}{\Gamma(9\alpha+1)(\Gamma(2\alpha+1))^2}\left[\frac{\Gamma(4\alpha+1)\Gamma(7\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)\Gamma(8\alpha+1)}+\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(5\alpha+1)}+\frac{\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)}\left\{\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2}{\Gamma(3\alpha+1)}\right\}\right]$

;$U_{1,\alpha}(2,10)=0$

and so on

Now, from (3.2), we have

$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{1,\alpha}(k,h)x^{1.k}t^{\alpha h}$ (3.6)

Using the above values of $U_{1,\alpha}(k,h)$ in (3.6), the solution of (1.1)is obtained as

$$u(x,t)=x^2+\frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}-\frac{4c(\Gamma(\alpha+1))^2\Gamma(4\alpha+1)}{(\Gamma(2\alpha+1))^2\Gamma(5\alpha+1)}t^{5\alpha}+\frac{16c^2(\Gamma(\alpha+1))^3\Gamma(4\alpha+1)\Gamma(7\alpha+1)}{(\Gamma(2\alpha+1))^3\Gamma(5\alpha+1)\Gamma(8\alpha+1)}t^{8\alpha}-\frac{4c\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}x^2t^{3\alpha}+\frac{8c^2(\Gamma(\alpha+1))^2\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)\Gamma(2\alpha+1)}\left[\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2}{\Gamma(3\alpha+1)}\right]x^2t^{6\alpha}-\frac{32c^3(\Gamma(\alpha+1))^3\Gamma(8\alpha+1)}{\Gamma(9\alpha+1)(\Gamma(2\alpha+1))^2}\left[\frac{\Gamma(4\alpha+1)\Gamma(7\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)\Gamma(8\alpha+1)}+\frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(5\alpha+1)}+\frac{\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)}\left\{\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)}+\frac{2}{\Gamma(3\alpha+1)}\right\}\right]x^2t^{9\alpha}+\cdots$$

(3.7)

Take $c=0$ in (1.1) and (3.7),the non-linear inhomogeneous time fractional partial differential equation becomes linear and its solution is

$u(x,t)=x^2+\frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}$ (3.8)

which is same as obtained by Z. Odibat and S. Momani [18]using Generalized differential transform method.

Conclusions

This present analysis exhibits the applicability ofthe generalized differential transform method to solvenon-linear inhomogeneous time-fractional partial differential equation. It may be concluded that GDTM is a reliabletechnique to handle linear and nonlinear fractional differential equations. This technique provides more realistic series solutions as compared with other approximate methods.

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