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# A Numerical Study of an Application of the Multi-parameter Local Bifurcation Theory

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## Abstract
In this study we deal with a fourth order boundary value problem to verify the bifurcation theory developed in Bari(2002). We study the bifurcation of the non-trivial solution in the neighborhood of trivial state .In so doing we verify the theoretical bifurcation- results as well as produce some numerical results of that problem using a Computer Algebra System (CAS), *Mathematica*.

**Keywords:** Boundary Value Problem, Bifurcating Points, Eigenvalue, Multi-parameter Nonlinear Eigenvalue Problem, Non-trivial solutions

## Introduction
In this paper we deal with the following boundary value problem

$$F(\lambda,x) \equiv L(0)x + \sum_{i=1}^{p} \lambda_i L_i x + f\big(t,x(t)\big) = 0, \quad on\ [0,\pi]$$
$$x'(0) = x'(\pi) = x'''(0) = x'''(\pi) = 0$$

(1.1)

for $P \geq 2$, where

$$L(0)x = x'''' + 5x'' + 4x,$$

$$L_i x = a_i x'' + b_i x' + c_i x, \quad i = 1, \ldots, p,$$

are linear operators defined on
$X = \{x \in C^4([0,\pi]) : x'(0) = x'(\pi) = x'''(0) = x'''(\pi) = 0\}$
with values in $Y = C([0,\pi])$ and $f(t,\xi) \in \mathbb{R}$ is a continuous function of $(t,\xi) \in \mathbb{R}^2$ and a $C^2$ function of $\xi$ for each fixed $t$, with
$f(t,0) = f_x(t,0) = 0.$

Note that $(\lambda, 0)$ is a solution of (1.1) for all $\lambda \in \mathbb{R}^p$. Such a solution is called the *trivial* solution. We will find the bifurcation of the non-trivial solutions (i.e. solutions $(\lambda, x)$ with $x \neq 0$) of equation (1.1) in a neighbourhood (nbd) of $(\lambda^0, 0) = (0,0)$. A point $\lambda^0 \in \mathbb{R}^p$ is called a *bifurcation point* of equation (1.1) from the trivial state $(\lambda, 0)$ if each nbd of $(\lambda^0, 0)$ contains non-trivial solutions.of (1:1). So by the Implicit Function Theorem (see Bari(1995)), a necessary condition for $\lambda = \lambda^0$ to be a bifurcation point of (1.1) is that
$\dim N\big(D_x F(\lambda^0, 0)\big) \geq 1,$

where $N$ denotes the null space of $D_x F(\lambda^0, 0) = L(0) + \sum_{i=1}^{p} \lambda_i^0 L_i.$

This problem was studied by Lopez-Gomez (1989) for $p = 2$. The following bifurcation theory for multi-parameter nonlinear eigenvalue problems is established in Bari(2002(1)).

*Theorem 1. Let* $\lambda^0$ *be a point in* $\mathbb{R}^p$, $p \geq 1$ *such that* $L(\lambda^0) = 0$ *and the condition*

(C) *for all* $\phi \in \mathbb{R}^p$ *with* $\|\phi\| = 1$, *span* $[L_1\phi, \dots, L_p\phi] = \mathbb{R}^m$

*holds. Then* $(\lambda^0, 0)$ *is a bifurcation point of the equation*

$$F(\lambda, u) \equiv L(\lambda)(u) + q(\lambda)(u) + h(\lambda, u)(u) = 0 \quad (1.2)$$

*where for each* $\lambda \in \mathbb{R}^p, L(\lambda)$ *is in* $L(\mathbb{R}^n, \mathbb{R}^m)$ *and is a* $C^{r-1}$ *function of* $\lambda$; $q: \mathbb{R}^p \to Q(\mathbb{R}^n, \mathbb{R}^m)$ *is a* $C^{r-2}$ *function of* $\lambda$, *while for each* $(\lambda, u)$, $h: \mathbb{R}^p \times \mathbb{R}^n \to Q(\mathbb{R}^n, \mathbb{R}^m)$ *is a function of* $(\lambda, u)$ *with* $h(\lambda, 0) = 0$. *Also there exists a unique* $C^{r-1}$ *mapping* $\mu: E \to \mathbb{R}^p$ *which is of class* $C^r$ *away from* $s = 0$, *defined in a neighbourhood* $U$ *of* $E_0 = \{0\} \times S^{\ell-1} \times \{0\}$ *in* $E = \{(s, \phi, v) \in \mathbb{R} \times S^{n-1} \times \mathbb{R}^p : (DL(\lambda^0)v)\phi = 0\}$ *such that*

$\mu(s, \phi, v) \in M(\phi) = \{\lambda \in \mathbb{R}^p : (DL(\lambda^0)\lambda)\phi \neq 0\}$, *for all* $(s, \phi, v) \in U$

*and the point*

$\big(\lambda(s, \phi, v), u(s, \phi)\big) := (\lambda^0 + \mu(s, \phi, v) + v, s\phi) \in \mathbb{R}^p \times \mathbb{R}^n$

*satisfies equation (1.2). In addition, all solutions of (1.2) near* $(\lambda^0, 0)$ *are of this form.*

It is shown in Bari[1] that the condition (C) holds 'generically' for $p \geq n + m - 1$ in the sense that the set of matrices $(L_1, \dots, L_p)$ satisfying (C) is open and dense in the set of all such collections; also for $n = m = p = 2$, roughly half of the pairs $(L_1, L_2) \in M(2,2)^2$ satisfy (C). In this study we will verify these results by example (1.1). For $p \geq 3$, we will verify the following theorem proved in Bari(1995):

*Theorem 2.* $\mathcal{M}_l$ is open in $\mathcal{M}^p$. Also it is dense in $\mathcal{M}^p$ if $p \geq n + m - 1$, where $\mathcal{M}$ and $\mathcal{M}_l$ denote the set of all collections $(L_1, \dots, L_p)$ which satisfy the spanning condition (C).

An outline of this paper is as follows: In Section 2 we verify the bifurcation results of Bari(2002(2)). In Section 3, we will give the numerical results of equation (1.1) for $p = 1$.

**Verification of the Bifurcation Results**

The null space and the range of $L(\lambda^0) = L(0) + \sum_{i=1}^{p} \lambda_i^0 L_i$ at the point $\lambda^0 = 0 \in \mathbb{R}^p$ are $N\big(L(0)\big) = \text{span}\,[\cos t, \cos 2t]$ and $R\big(L(0)\big) = \{v \in C(0, \pi) : \int_0^{\pi} v(t) \cos t \, dt = \int_0^{\pi} v(t) \cos 2t \, dt = 0\}$ respectively. Also

$$L_i \cos t = (c_i - a_i)\cos t - b_i \sin t, \quad i = 1, \dots, p,$$

$$L_i \cos 2t = (c_i - 4a_i)\cos 2t - 2b_i \sin 2t, \quad i = 1, \dots, p.$$

Let $P$ be the projection of $Y$ onto a complement of $R\big(L(\lambda^0)\big)$ given by

$$P_u = \frac{\int_0^{\pi} u(t) \cos t \, dt}{\int_0^{\pi} \cos^2 t \, dt} \cos t + \frac{\int_0^{\pi} u(t) \cos 2t \, dt}{\int_0^{\pi} \cos^2 2t \, dt} \cos 2t.$$

We need to show:

$$\text{span}\,[PL_1(\alpha \cos t + \beta \cos 2t), \dots, PL_p(\alpha \cos t + \beta \cos 2t)] \oplus R\big(L(\lambda^0)\big) = Y$$

for all $(\alpha, \beta)$ with $\alpha^2 + \beta^2 = 1$.

Now

$$L_i(\alpha \cos t + \beta \cos 2t) = \alpha[(c_i - a_i)\cos t - b_i \sin t] + \beta[(c_i - a_i)\cos 2t - 2b_i \sin 2t].$$

Since

$$P(\sin t) = -\frac{4}{3\pi}\cos 2t, P(\sin 2t) = \frac{8}{3\pi}\cos t,$$

we have

$$PL_i(\alpha \cos t + \beta \cos 2t) = \left[\alpha(c_i - a_i) - \frac{16}{3\pi}\beta b_i\right]\cos t + \left[\frac{4}{3\pi}\alpha b_i + \beta(c_i - 4a_i)\right]\cos 2t$$

First of all we consider the case $p = 2$. In this case we need to show that the elements $PL_i \quad (\alpha \cos t + \beta \cos 2t), i = 1,2$ of the complements of $R(L(0))$ are linearly independent for all $(\alpha, \beta) \in S^1$.

Now for $\lambda_i \in \mathbb{R}$,

$$\sum_{i=1}^{2} \lambda_i PL_i(\alpha \cos t + \beta \cos 2t) = 0$$

$$\Rightarrow \sum_{i=1}^{2} \lambda_i \left[ \left( \alpha(c_i - a_i) - \frac{16}{3\pi} \beta b_i \right) \cos t + \left( \frac{4}{3\pi} \alpha b_i + \beta (c_i = 4a_i) \right) \cos 2t \right] = 0$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{2} \lambda_i \left[ \alpha(c_i - a_i) - \frac{16}{3 <} \beta b_{<} \right] = 0 \\ \sum_{i=1}^{2} \lambda_i \left[ \frac{4}{3\pi} \alpha b_i + \beta (c_i - 4a_i) \right] = 0 \end{cases}$$

$$\begin{bmatrix} \alpha(c_1 - a_1) - \frac{16}{3\pi}\beta b_1 & \alpha(c_2 - a_2) - \frac{16}{3\pi}\beta b_2 \\ \frac{4}{3\pi}\alpha b_1 + \beta(c_1 - 4a_1) & \frac{4}{3\pi}\alpha b_2 + \beta(c_2 - 4a_2) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For linearly independent we need

$$\begin{vmatrix} \alpha(c_1 - a_1) - \frac{16}{3\pi}\beta b_1 & \alpha(c_2 - a_2) - \frac{16}{3\pi}\beta b_2 \\ \frac{4}{3\pi}\alpha b_1 + \beta(c_1 - 4a_1) & \frac{4}{3\pi}\alpha b_2 + \beta(c_2 - 4a_2) \end{vmatrix} \neq 0, \quad \forall (\alpha, \beta) \in S^1$$

i.e., we need

$$\frac{4}{3\pi} \alpha^2 [b_2(c_1 - a_1) - b_1(<_2 - <_2)] + \alpha\beta[(c_1 - a_1)(c_2 - 4a_2) - (c_1 - 4a_1)(c_2 - a_2)]$$

$$+ \frac{16}{3\pi} \beta^2 [b_2(c_1 - 4a_1) - b_1(c_2 - 4a_2)] \neq 0, \quad \forall (\alpha, \beta) \in S^1 \qquad (2.1)$$

Let us define,

$$A = 4/3\pi \, [b_2 (c_1 - a_1) - b_1 (c_2 - a_2)]$$

$$B = \frac{1}{2} [(c_1 - a_1)(c_2 - 4a_2) - (c_1 - 4a_1)( c_2 - a_2)]$$

$$C = \frac{16}{3\pi} [b_2(c_1 - 4a_1) = b_1(c_2 - 4a_2)]$$

Then (2.1) becomes,

$$A\alpha^2 + 2B\alpha\beta + C\beta^2 \neq 0$$

$$\Rightarrow (\alpha \quad \beta) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0, \forall (\alpha, \beta) \in S^1$$

$$\Rightarrow \begin{pmatrix} A & B \\ B & C \end{pmatrix} \text{ is positive or negative definite}$$

$$\Rightarrow AC - B^2 > 0$$

So for $n = m = p = 2$, the spanning condition can old in this example for 'half' of the matrices in the matrix space. Returning to the coefficient space we find, for example, if $a_i = b_1 = 1, b_2 = 2, c_i = 5,$ for $i = 1, 2$ then we get $A = C = \frac{16}{3\pi}, B = 0,$ which satisfy the spanning condition. Therefore there exist a non-empty, open set of coefficients which satisfies the condition. But if we take $b_i = 0$ then the condition (C) does not hold for any $a_i, c_i, i = 1, 2$. Also the points $b_i = 0$ and $a_i, c_i$ are such that $B_i \neq 0$ is an interior point. So there exists a closed set, with non-empty interior, not satisfying the condition.

So for 𝒳 $= 2$, giving conditions on $a_i, b_i, c_i, i = 1,2$, we can obtain bifurcation for $(\lambda_1, \lambda_2, x) = (0,0,0)$ by applying Theorem 1. We will use these results in the nextsection to get the non-trivial solution of equation (1.1) using a CAS (Mathematica(1999)).

Now suppose, $p \geq 3$. We want to show that the spanning condition holds generically. But it is difficult to calculate explicitly. So we will check whether the matrices

$$M_i = \begin{pmatrix} c_i - a_i & -\frac{16}{3\pi} b_i \\ \frac{4}{3\pi} b_i & c_i - 4a_i \end{pmatrix}, \qquad i = 1, \dots, p,$$

representing the differential operator $PL_i$ satisfy the condition,

(M) given any $\phi \in S^{n-1}, \psi \in \mathbb{R}^{<}, \exists\, M \in \mathcal{M}$ such that $M\phi = \psi$

or not. So take any $\phi \in (\phi_1, \phi_2) \in S^1$ and $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$, then

$$\begin{pmatrix} c_i - a_i & -\frac{16}{3\pi} b_i \\ \frac{4}{3\pi} b_i & c_i - 4a_i \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} (c_i - a_i)\phi_1 - \frac{16}{3\pi} \square_i \phi_2 = \psi_1 \\ \frac{4}{3\pi} b_i \phi_1 + (c_i - 4a_i)\phi_2 = \psi_2 \end{cases}$$

So for appropriate values of $a_i, b_i, c_i, i = 1,2$, the class of matrices $M_i$ satisfy the condition (M). Hence by Theorem 2 the spanning condition holds generically for $p \geq 3$. Hence Theorem 1 is applicable.

**Numerical Results**

Letting $p = 2$ and substituting the values of $a_i = b_1 = 1, b_2 = 2$ and $c_i = 5$ for $i = 1,2$ in (1.1) we get the following differential equation

$$x''''(t) + (5+\lambda_1+\lambda_2)x''(t) + (\lambda_1+2\lambda_2)x'(t) + (4+5\lambda_1+5\lambda_2)x = 0 \quad (3.1)$$

(For simplicity we have taken $f(t, x(t)) = 0$). As it is not possible to solve the equation (1.1) directly by MATHEMATICA, we use the following techniques to get the solution of the equation (3.1):

Let $x(t) = e^{mt}$ be the solution of (3.1). Then for fixed $\lambda_1 = 0, \lambda_2 = \lambda,$ we get the auxiliary equation of (3:1) as

$$m^4 + (5+\lambda)m^2 + 2\lambda m + (4+5\lambda) = 0 \quad (3.2)$$

Then we use the following steps in MATHEMATICA:

```
bifurc[K5_, K6_, K7_, K8_, xrange_, yrange_] := Module[{},
Clear[m, r , m1, m2, m3, m4, a, b, c, d, k1, k2, k3, k4, k5, k6,
k7, k8, K5, K6, K7, K8, n, tt, t, kk1, kk2, kk3, kk4];
result = Solve[m^4 + (5 + r)*m^2 + 2*r*m + 4 + 5 r == 0, m];
a[r_] := 1/2 Sqrt[-5 - r + (5 + r)/3 + (73 + 70 r + r^2)/
(3 (-595 - 969 r - 111 r^2 + r^3 + 6 Sqrt[3] Sqrt[-324
+ 315 r - 167 r^2 - 1479 r^3 - 42 r^4 - 4 r^5])^(1/3))];
b[r_] := 1/2 Sqrt[-5 + (1/3) (-5 - r) - r - (73 + 70 r + r^2)/
(3 (-595 - 969 r - 111 r^2 + r^3 + 6 Sqrt[3] Sqrt[-324
+ 315 r - 167 r^2 - 1479 r^3 - 42 r^4 - 4 r^5])^(1/3))];
c[r_] := 1/3 (-595 - 969 r - 111 r^2 + r^3 + 6 Sqrt[3] Sqrt[-324 + 315 r - 167 r^2
- 1479 r^3 - 42 r^4 - 4 r^5])^(1/3);
d[r_] := (4 r)/(Sqrt[-5 - r + (5 + r)/3 + (73 + 70 r + r^2)/
(3 (-595 - 969 r - 111 r^2 + r^3 + 6 Sqrt[3] Sqrt[-324
+ 315 r - 167 r^2 - 1479 r^3 - 42 r^4 - 4 r^5])^(1/3)) +
1/3 (-595 - 969 r - 111 r^2 + r^3 + 6 Sqrt[3] Sqrt[-324
+ 315 r - 167 r^2 - 1479 r^3 - 42 r^4 - 4 r^5])^(1/3)]);
m1[r_] := a[r] - b[r] - c[r] - d[r];
m2[r_] := a[r] + b[r] - c[r] - d[r];
m4[r_] := -a[r] + b[r] - c[r] + d[r];
m3[r_] := -a[r] - b[r] - c[r] + d[r];
x[r_, t_] :=
Exp[Re[m1[r]]*t]*(k1*Cos[Im[m1[r]]*t]
+ k2*Sin[Im[m1[r]]*t]) +
Exp[Re[m2[r]]*t]*(k3*Cos[Im[m2[r]]*t]
+ k4*Sin[Im[m2[r]]*t]) +
Exp[Re[m3[r]]*t]*(k5*Cos[Im[m3[r]]*t]
+ k6*Sin[Im[m3[r]]*t]) +
Exp[Re[m4[r]]*t]*(k7*Cos[Im[m4[r]]*t]
+ k8*Sin[Im[m4[r]]*t]);
f[t_] := x[0, t] // N;
eqn1 = D[f[t], t] /. t -> 0;
eqn2 = D[f[t], t] /. t -> Pi;
eqn3 = D[f[t], {t, 3}] /. t -> 0;
eqn4 = D[f[t], {t, 3}] /. t -> Pi;
soln = Solve[{eqn1 == 0, eqn2 == 0, eqn3 == 0, eqn4 == 0}, {k1, k2,
k3, k4, k5, k6, k7, k8}];
intlst = {k5 -> K5, k6 -> K6, k7 -> K7, k8 -> K8};
kk1 = soln[[1, 1, 2]] /. intlst;
kk2 = soln[[1, 2, 2]] /. intlst;
kk3 = soln[[1, 3, 2]] /. intlst;
kk4 = soln[[1, 4, 2]] /. intlst;
k1 = kk1; k2 = kk2; k3 = kk3; k4 = kk4;
k5 = K5; k6 = K6; k7 = K7;
k8 = K8;
```

```
tab1 = Table[{r, x[r, 0]}, {r, -0.1, 0.1, 0.01}];
tab2 = Table[{r, x[r, 1]}, {r, -0.1, 0.1, 0.01}];
tab3 = Table[{r, x[r, 2]}, {r, -0.1, 0.1, 0.01}];
tab4 = Table[{r, x[r, 3]}, {r, -0.1, 0.1, 0.01}];
q1 = ListPlot[tab1, Joined -> True,
PlotStyle -> {Red, Thick},
DisplayFunction -> Identity];
q2 = ListPlot[tab2, Joined -> True,
PlotStyle -> {Green, Thick},
DisplayFunction -> Identity];
q3 = ListPlot[tab3, Joined -> True,
PlotStyle -> {Blue, Thick},
DisplayFunction -> Identity];
q4 = ListPlot[tab4, Joined -> True,
PlotStyle -> {Orange, Thick},
DisplayFunction -> Identity];
qq2 = Show[q1, q2, q3, q4,
PlotRange -> {{-xrange, xrange}, {-yrange, yrange}},
Frame -> True, AxesLabel -> {"r", "x"}]
]
```

Choosing the two set values for the above coded function '**bifurc**', we get the following graph of $x(\lambda, t)$ in a neighbourhood of $(\lambda^0 = 0)$ for the same values of $t$:

```
qq1 = bifurc[1, 1, 1, 1, 0.005, 100]
qq2 = bifurc[1, -1, 1, 0, 0.005, 100]
```

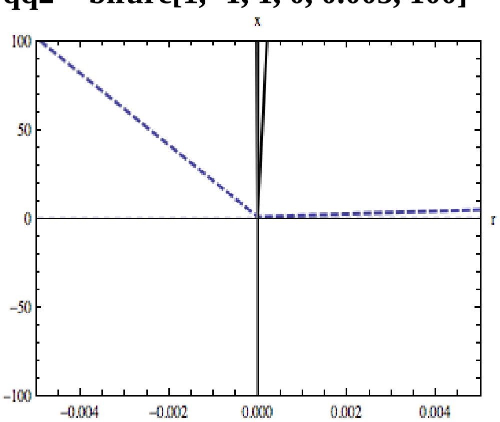


**Fig. 1: Graph of** $x(\lambda, t)$ **in a neighbourhood of** $\lambda^0 = 0$ **for** $t = 0,1,2,3$.

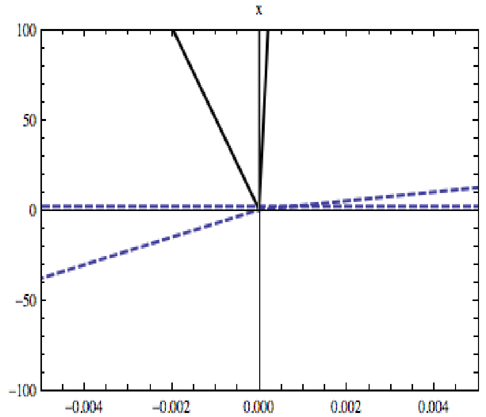


**Fig. 2: Graph of** $x(\lambda, t)$ **in a neighbourhood of** $\lambda^0 = 0$ **for** $t = 0,1,2,3$.

## Conclusion

Numerical results presented in this paper suggest that the bifurcation of the non-trivial solution in the neighborhood of the trivial state exists in $\mathbb{R}^2$. Alongside this numerical presentation a verification of the bifurcation result is also obtained. Some indicative results are also obtained for three dimensional spanning. *Mathematica* module developed in this paper can be used in further extension of the similar numerical works for higher dimensional case; which is left as a future work.

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