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Fundamental group of Heisenberg group

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Abstract

Fundamental groups of Topological Spaces have an important role in Algebraic Topology as they are useful in study of properties of the Topological Spaces. Therefore, this paper makes an attempt to construct the Fundamental group of an interesting and important algebraic structure, the Heisenberg group. This construction is made by using some interesting properties of Heisenberg group. Properties of matrix Lie groups are also used as Heisenberg is also a matrix Lie group.

Keywords: Heisenberg Group, Fundamental group, Lie group

Introduction

As Heisenberg group is a matrix Lie group, it is necessary to go through some of the basics of Lie groups and Matrix groups. All the notations and terminology used in this paper are referred to [5]. It is well known that the set of all $n \times n$ invertible matrices over complex field $\mathbb{C}$ form a group with respect to matrix multiplication and is denoted by $GL(n;\mathbb{C})$. Since $\mathbb{R} \subset \mathbb{C}$, it is easy to see that $GL(n;\mathbb{R})$ is a subgroup of $GL(n;\mathbb{C})$, where $GL(n;\mathbb{R})$ is the group of all $n \times n$ invertible matrices over real numbers field. A subgroup G of $GL(n;\mathbb{C})$ is a matrix Lie group if A_m is any sequence of matrices in G, and A_m converges to some matrix A then either $A \in G$, or A is not invertible. It means to say that G is closed subset of $GL(n;\mathbb{C})$. An element a in a ring $(R,+,\cdot)$ is said to be nilpotent if there exists a positive integer 'n' such that $a^n = 0$. The exponential of a real or complex matrix X is denoted by e^X and is defined by $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$.

Before going to the definition of Fundamental group of a topological space, it is necessary to go through some basics. A mapping f from a topological space X into a topological space Y is said to be continuous if the inverse image $f^{-1}(V)$ of each open set V of Y is open in X. A path in a topological space X is a continuous map $f: I \to X$, where I is the unit interval. If the beginning and end points of a path coincide in a topological space, then the path is known as a loop. A homotopy of paths in X is a family $f_t: I \to X$, $0 \le t \le 1$, such that the end points of the path are independent of t and the associated map $F(s,t): I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous. When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be homotopic and is denoted by $f_0 \simeq f_1$. It is easy to see that the relation of homotopy on paths with fixed endpoints in any topological space is an equivalence relation and the equivalence class of a path f is denoted by [f]. If the starting and end points of a path are the same, then the path is known as a loop, and this common starting and ending point is referred to as the base point and is denoted by x_0 . Given two paths $f, g: I \to X$ such that $f(1) = g(0)$, the product path $f.g$ that traverses first f and then g, defined by $f.g = \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & 1/2 \le s \le 1 \end{cases}$. The set of all homotopy classes [f] of loops $f: I \to X$, at the base point x_0 is denoted by $\pi_1(X, x_0)$. The set $\pi_1(X, x_0)$, forms a group with respect to the product defined by [f] [g] = [f . g], is called the 'Fundamental Group' of X at the base point x_0.

Main Results

In this section, we constructed first the Heisenberg group and then its Fundamental Group. To construct the Fundamental Group the properties of Heisenberg group have been used. It has many applications in Quantum mechanics and Wave mechanics. Throughout this paper, the Heisenberg group has been considered over the real numbers field only. It is obvious that it can be defined over the field of Complex numbers also.

Proposition 1: The set H of all 3×3 matrices A of the form $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, where a, b, c are arbitrary real numbers, forms a group with respect to multiplication of matrices.

Proof: Since the product of two upper triangular matrices is again an upper triangular, it is easy to see that for all A, B in $H, AB \in H$. It is easy to see that Associative law with respect to multiplication holds in H, as multiplication of matrices is associative. Since $A.I_3 = I_3 A = A$, it is clear that the matrix $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the Identity element in H. To compute the inverse of A, let $B = \begin{bmatrix} d & e & f \\ p & q & r \\ x & y & z \end{bmatrix}$ be any 3×3 matrix. Consider the matrix equation $AB = I_3$.

$$\Rightarrow \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ p & q & r \\ x & y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow d = 1, e = -a, f = ac - b, p = 0,$$
$$q = 1, r = -c, x = 0, y = 0, z = 1.$$

Therefore, $B = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$, is the inverse of the matrix A. Since B is also a 3×3 upper triangular matrix and in the form of the matrix A, it belongs to H . Hence, H forms a group with respect to matrix multiplication.

Definition 2: The set H of all 3×3 matrices A of the form $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, where a, b, c are arbitrary real numbers, forms a group with respect to matrix multiplication is known as "Heisenberg group".

Since matrix multiplication is not commutative, Heisenberg group is not Abelian.

Towards properties of Heisenberg group we have

Lemma 3: Heisenberg group H is a matrix Lie group.

Proof: If A_m is any sequence of matrices in H , then it is easy to see that the limit of the sequence A_m is again an upper triangular matrix of the form of matrix A. Hence, H is a matrix Lie group.

Definition 4: A matrix Lie group is G is said to be 'compact' if

(i) A_m is any sequence of matrices in $G,$ and A_m converges to a matrix $A,$ then A is in G and

(ii) there exists a constant k such that for all $A \in G, \left|A_{ij}\right| \le k$ for all $1 \le i, j \le n.$

Lemma 5: Heisenberg group is a non-compact matrix Lie group.

Proof: Although Heisenberg group satisfies condition (i) in definition 4, it fails to satisfy condition (ii) of definition 4 because the determinant value of every upper triangular matrix need not be the same. Hence, Heisenberg group is non-compact matrix Lie group..

Definition 6: A matrix Lie group G is said to be connected if for any two given matrices A and B in $G,$ there exists a continuous path $F(t), a \le t \le b,$ lying completely in G with $F(a) = A$ and $F(b) = B.$

Lemma 7: Heisenberg group is path connected.

Proof: Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ be any two elements in the Heisenberg group H.

Therefore, $a,b,c,x,y,z \in \mathbb{R}.$ Denote the elements A and B by the relations $f(a,b,c)$ and $f(x,y,z)$ respectively. Define a function

$F(t) = f(a + (x-a)t, b + (y-b)t, c + (z-c)t),$ $t \in [0,1].$

It is clear that $F(0) = f(a,b,c) = A$ for $t = 0$ and $F(1) = f(x,y,z) = B$ for $t = 1.$ Therefore, for each value of t between 0 and 1 we get a continuous path from A to $B.$ It is obvious to see that this path completely lies in the group H as $F(t) \in H \ \forall t \in [0,1].$ Hence, the Heisenberg group H is path connected.

The above lemma can also be proved by using the properties of Lie algebra of Heisenberg group. Since our aim is to construct the Fundamental group of Heisenberg group, it is not being considered here.

Definition 7: A mapping f from a topological space X in to a topological space Y is said to be a homeomorphism if (i) f is a bijection, (ii) both f, f^{-1} are continuous.

Definition 8: A matrix Lie group G is said to be simply connected if (i) it is connected, and

(ii) every loop in G can be shrunk continuously to a point in $G.$

Example: The Euclidean space $\mathbb{R}^n$ is simply connected for all positive integers $n.$

Theorem 8: The Heisenberg group H is homeomorphic to the space $\mathbb{R}^3$ and hence it is simply connected.

Prrof: Let $f : \mathbb{R}^3 \to H$ be a mapping defined by $f(x,y,z) = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ for $x,y,z \in \mathbb{R}..$ To prove f is well defined and one-one suppose $f(x,y,z) = f(a,b,c)$ for any $(x,y,z),(a,b,c) \in \mathbb{R}^3.$

$$\Leftrightarrow \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow (x,y,z) = (a,b,c) \Leftrightarrow f$$

is well defined and one-one.

To prove that f is onto, let $\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in H$ be any element. Therefore, $x, y, z \in \mathbb{R}$. This implies $(x, y, z) \in \mathbb{R}^3 \Rightarrow f$ is onto. Thus, f is a bijection and hence f^{-1} is also a bijection. Also, it is easy to see that f and f^{-1} are continuous and hence f is a homeomorphism. Since $\mathbb{R}^3$ is simply connected and hence the Heisenberg group H.

Theorem 9: If X and Y are path connected homeomorphic spaces, then their Fundamental groups are isomorphic.

Proof: See Theorem 2.4.6 in [6].

Theorem 10: The Fundamental group of the Heisenberg group H is isomorphic to the Fundamental group of $\mathbb{R}^3$. That is, $\pi_1(H) \cong \pi_1(\mathbb{R}^3)$.

Proof: In view of theorems 8 and 9, the proof follows.

Theorem 11: The Fundamental group of the Heisenberg group is the trivial one. That is, $\pi_1(H) = \{1\}$, where 1 is the multiplicative identity.

Proof: Since the Fundamental group of the space $\mathbb{R}^3$ is the trivial one, in view of the theorem 10, the proof follows.

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