International Journal of Sciences & Applied Research

www.ijsar.in

ON SOME PROPERTIES OF TRANSFORMATION GRAPHS

V. Ravi and P. Jagadeeswari

Department of Mathematics, Bharath Institute of Higher Education and Research, Faculty of Arts and Science, Chennai, INDIA

ABSTRACT

In this paper, we present basic properties, covering invariants, connectivity and characterization of graphs whose transformation graph G+−+ are eulerian.

1 INTRODUCTION

The graphs considered in this paper are finite, undirected and have no loops or multiple edges. The distance dG(u, v) between two vertices u and v in G is the length of shortest path joining them if any, otherwise dG(u, v) = ∞. The eccentricity ecc G(v) of ve tex v ∈ V (G), is the largest distance between v and all other vertices of G, that is,

$$eccG(v) = \max \{dG(u, v)/u \in V(G)\}.$$

The triangle is the cycle with three vertices.

The girth g(G) of the graph G , is the length of shortest cycle (if any) in G . The edge connectivity (G) of G, is defined to be the largest integer k for which G is k edge connected. Edge independent set is set of edges of G , such that no two of them are adjacent in G . Let a, b be any positive integer, 1 ≤ a ≤ b . A graph G is said to be (a, b) biregular or simply biregular if its vertices have degree either a or b.

The most interesting operation by which one graph is obtained from other is line graph. A line graph is denoted by L(G) of a graph G , is the graph with vertex set as the edge set of G and two vertices of L(G) are adjacent whenever the corresponding edges in G have a vertex in common. The jump graph J(G) of a graph G is a graph whose vertex set is the edge set of G and two vertices of J(G) are adjacent.

Corresponding edges in G are not adjacent in G. The jump graph J(G) of a graph G is the complement of the line graph L(G) of a graph G . The block graph B(G) of a graph G is a graph with vertex set as a set of all blocks of G and two vertices of B(G) are adjacent if and only if the corresponding blocks have the common vertex in G .

The transformation graph G+−+ of G is the graph with vertex set V (G) ∪ E(G) in which the
vertex x and y are joined by an edge if one of the following conditions holds:

- x, y ∈ V (G) and x and y are adjacent in G,
- x, y ∈ E(G) and x and y are not adjacent in G,
- one of x and y is in V (G) and the other is in E(G), and they are incident in G.

In G+−+ the vertices corresponding to the vertices (edges) in G are called as point (line) vertices.

The following is the simple algorithm to construct the transformation Graph G+−+ of G.

- On each edge e = uv of a graph G , draw a triangle with vertices u , v and v(e) , where v(e) is the new vertex introduced by an edge e of G in G +−+.
- Draw an edge v(ei)v(ej) in G+−+ , if ei and ej are not adjacent in G.

In Figure 1 the graph G and simple algorithm to construct transformation graph G+−+ are shown.

The following results will be useful in the proof of our results.

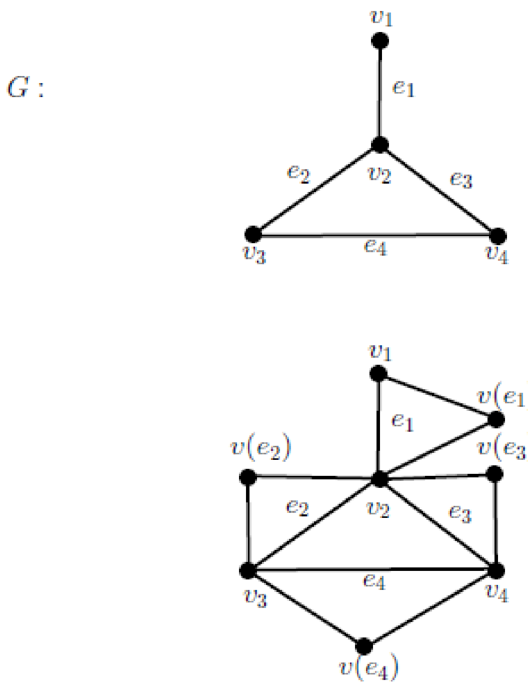
Theorem A. Let G be a (p, q) graph, then

(i) J(G) is a graph, where di is the degree of the vertex vi in G.
(ii) $\Delta(J(G)) = q + 1 - \delta'(G)$, where $\delta'(G)$ is the minimum edge degree of G.
(iii) $\delta(J(G)) = q + 1 - \Delta'(G)$, where $\Delta'(G)$ is the maximum edge degree of G.

Theorem 2.B. The transformation graph G+−+ is connected if and only if G contains no isolated vertices.

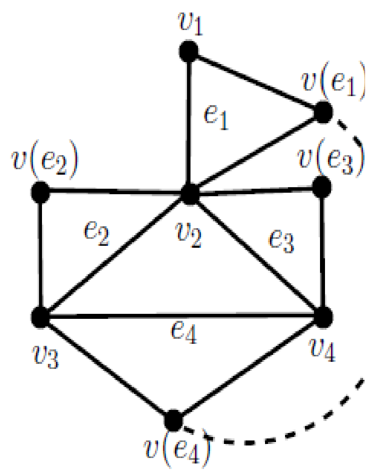
Theorem 2.C. The graph G is eulerian if and only if every vertex of G has even degree.

Remark 2.1. Let G be a (p, q) graph, and H = G +−+ . For every vertex v ∈ V(G), deg H(v) = 2 deg G(v) and for any e ∈ E(G), then degH (e) =2+ r, where r is the number of edges not adjacent to e in G.



Step- 1

G^{+-+} :



Step- 2

Figure 2.1

Remark. For any graph G, G and J(G) are the edge disjoint subgraphs of G^{+-+} and subdivision graph S(G) is the spanning subgraph of G^{+-+} .

2 Main Results

Theorem 2. Let G be a (p, q) graph whose vertices have degree di. Then

and

Proof. By the definition of G^{+-+} , $|V(G^{+-+})| = p + q$. By the algorithm we observe that an edge e = uv in G , introduces a vertex v(e) in G^{+-+} , and forms a triangle, which contributes 3q edges and further the edges of jump graph are added.

Therefore by Theorem 2.A.

where di is the degree of the vertex vi in G

Theorem 2.2 Let G be a (p, q) graph and H = G^{+-+} . Then

$\Delta(H) = \max\{2\Delta(G), 3 + q - \delta'(G)\}$

and $\delta(H) = \min\{2\,\delta(G), 3 + q - \Delta'(G)\}$.

Proof. By the definition of the transformation graph G^{+-+} of G, $V(H) = V(G) \cup E(G)$. We consider the following cases:

Case 1. If a vertex $u \in V(G)$, then by Remark 2.1, $\deg_H(u) = 2\deg_G(u)$, therefore $\Delta(H) = 2\Delta(G)$ and $\delta(H) = 2\,\delta(G)$.

Case 2. If an edge $e \in E(G)$, then by Remark 2.1, $\deg_H(e) = 1 + q -$ (edge degree of e in G)-2=3+ q -(edge degree of e in G), therefore $\Delta(H) = 3 + q - '(G)$ and $(H) = 3 + q - '(G)$. Hence from the above cases we get the result.

Theorem 2.3 Let G be n -regular graph with p vertices and H = G^{+-+} . Then H is biregular graph with p point vertices of degree 2n and (np)/2 line vertices of degree $3 + (n(p-4))/2$.

Proof. If G is a n -regular graph with p vertices then G has (np)/2 edges. By Remark 2.1, each vertex of H corresponding to the vertex of G is of degree 2n and each edge in G is not adjacent with $(np)/2 - (2n - 1)$ edges in G . Hence by Remark 2.1, each vertex of H corresponding to edge of G is of degree $2 + (np)/2 - (2n - 1) = 3 + (n/2)(p - 4)$.

Theorem 2.4 For any (p, q) graph G , the number of triangles in G^{+-+} is q + s + t . Where t is the number of triangles and s is the number of edge independent set with cardinality 3 in G.

Proof. In the transformation graph G^{+-+} , we observe that there are three types of triangles; (i) Triangles whose vertices belongs to $V(G)$, clearly it is t , (ii) triangles whose vertices belongs to $E(G)$ and the edges which are not adjacent in G are the vertices adjacent in G^{+-+} , clearly it is s , (iii) triangles whose one vertex belongs to $E(G)$ and other two belongs $V(G)$, by the algorithm of construction of G^{+-+} , each edge in G has triangle in G^{+-+} , clearly it is q

Theorem 2.5 Let G be any graph with at least one edge. Then girth $g(G^{+-+}) = 3$.

Proof. For any graph G with at least one edge, by algorithm of construction of the transformation graph G^{+-+} , each edge has a triangle in G^{+-+} . Hence $g(G^{+-+}) = 3$.

Theorem 2.6 Let G be a connected graph, and $H = G^{+-+}$. Then $ecc_H(v) \leq 3$ and equality holds if $u, v \in V(G)$, $d_G(u, v) \geq 3$.

Proof. Let G be a connected graph, then by definition of G^{+-+} , $V(G^{+-+}) = V(G) \cup E(G)$. We consider the following cases.

Case 1. Let u_i and $u_j \in V(G)$.

Subcase 1.1. If u_i and u_j are adjacent in G , then they are also adjacent in H , clearly $d_H(u_i, u_j) = 1$.

Subcase 1.2. If u_i and u_j are not adjacent in G . We have the following subcases.

Subcase 1.2.1. Suppose $d_G(u_i, u_j) = 2$, then clearly $d_H(u_i, u_j) = 2$.

Subcase 1.2.2. Suppose dG(ui, uj) > 2 . Then the edges incident with u_i and u_j are not adjacent in G (say e_i and e_j respectively). Therefore by definition of H, e_i and e_j are adjacent in H . Hence there exists a path $u_i e_i e_j u_j$ in H , thus $d_H(u_i, u_j) = 3$.

Case 2. Let e_i and $e_j \in E(G)$, which are the vertices of H .

Subcase 2.1. Suppose e_i and e_j are adjacent in G and u is the vertex incident to e_i and e_j . Then there exists a path $e_i u e_j$ in H . Hence $d_H(e_i, e_j) = 2$.

Subcase 2.2. Suppose e_i and e_j are not adjacent in G , then they are adjacent in H . Therefore $d_H(e_i, e_j) = 1$.

Case 3. Let $u \in V(G)$ and $e \in E(G)$. We consider the following sub cases.

Subcase 3.1. Suppose u is incident with e in G then u is adjacent to in H. Hence $D_h(u, e) = 1$.

Subcase 3.2. Suppose u is not incident with e in G. We have the following subcases.

Subcase 3.2.1. Suppose u is adjacent to $v \in V(G)$ and e incident to v i.e. v is common to u and e in G . Then by definition of H , there exists a path uve in H. Thus $d_H(u, e) = 2$.

Subcase 3.2.2. Suppose there is no common vertex which is adjacent to u and incident to e in G . Then an edge (say e_i) incident to u is not adjacent to e in G. By definition of H , e_i is adjacent to e in H . clearly $d_H(u, e) = 2$

Theorem 2.7 For any (p, q) graph G with no isolates

$$\alpha_0(G^{+-+}) = \min \{p + \alpha_0(J(G)), q + \alpha_0(G)\} .$$

Proof. Let $G = (p, q)$ be graph with no isolates. By Remark 2.2, G and $J(G)$ are edge disjoint subgraphs of G^{+-+} . Then there are two kinds of covering sets of G^{++} .

The first kind is the set consisting of $V(G)$, which covers all the edges of G as well as $S(G)$ in G^{+-+} and the remaining edges of $J(G)$ are covered by $o(J(G))$. Thus $o(G^{+-+}) = p + o(J(G))$. The second kind is the set consisting of $E(G)$, which covers all the edges of $J(G)$ and the edges of $S(G)$, and the remaining edges of G are covered by $0(G)$. Thus α0 (G+−+) = q + α0 (G). α0 (G+−+) = min {p + α0 (J (G)), q + α 0(G)}.

Theorem 2.8 Let G ≠ 2K2 be a graph with no isolates and H = G+−+ . Then (H) = 2 if G has a pendant edge or an edge e in G is adjacent to every edge of G

Proof. Let G ≠2K2 be a graph with no isolates. If λ(G) = 0 , then clearly λ(H) > 1 . Suppose λ(G) ≥ 1 . Since every edge of G has a triangle in H , and it has no bridge. Hence λ(G) > 1 .

Suppose G has a pendant edge then G has a pendant vertex, whose degree is 2 in H. By removing the edges which are incident to pendant vertex H becomes disconnected.

Hence λ(H) = 2. Suppose e is an edge of G which is adjacent to every edge of G . Then the vertex e in H is adjacent to only the vertices which are incident to in G . Hence the degree of e is less than the other vertices of H , then by removing these edges which are adjacent to e in H , results into a Disconnected graph H . Thus λ(H) = 2 .

Corollary 2.1 If G = 2K2 , then λ(G+−+) = 1 .

Theorem 2.9 If G is a cycle Cp and H = G+−+ , then (H) ≤ 4 and equality holds for p ≥ 5 .

Proof. Let G be a cycle Cp for any integer p . Then δ(G) = 2 and Δ′(G) = 4, by Theorem 2.2, δ(H) = min {4, p − 1} . Here for p ≥ 5 , p −1 ≥ 4 . Hence δ(H) = 4 . But by Whitney's result we have λ(H) ≤ δ (H) . Thus λ (H) ≤ 4 . Next, any cycle is 2-regular and by Theorem 2.3, H is biregular with p point vertices of degree 4 and p line vertices of degree (3 + (p − 4)) = (p − 1) in H , hence for p ≥ 5 δ(H) = 4 . Therefore by removing these four edges which are incident to any point vertex of H , results into a disconnected graph. Hence λ(H) = 4 for p ≥ 5 .

Corollary 2.2 Let H= G^{+-+},then

$$\lambda(H)=\begin{cases}2 & if \quad G=C_3\\ 3 & if \quad G=C_4\end{cases}.$$

Theorem 2.10 The transformation graph G+−+ is eulerian if and only if G contains no isolates and each edge of G is non-adjacent with even number of edges in G.

Proof. Suppose the graph G+−+ is eulerian. Then every vertex of G+−+ is of even degree and G+−+ is connected. Hence by Theorem 2.B. G has no isolates. Suppose an edge e is non-adjacent with odd number of edges in G . Then by Remark 2.1., degH (e) is odd, a contradiction. Conversely, suppose G contains no isolates and each edge of G is non- adjacent with even number of edges in G . Then by Theorem 2.B., G+−+ is connected and by Remark 2.1. every vertex of G+−+ is of even degree. Therefore by Theorem 2.C, G+−+ is eulerian.

Proposition 2.1 If G is a star, then G+−+ is a block graph.

Proof. It is well known that a graph is block graph if and only if every block of it is complete. If G is a star then every edge of G is a block and hence every block of G+−+ is a triangle. Therefore G+−+ is a block graph.

Theorem 2.11 Let G be any connected (p, q) graph. Then G is a star if and only if B(G+−+) = L(G) .

Proof. Let G be a star. Then every edge of G is a block and hence every block of G+−+ is a triangle. The number of triangles in G+−+ is equal to the number of edges in G . The edges of G and the blocks of G+−+ are one to one correspondence in such a way that two edges of G are adjacent if and only if the corresponding block the corresponding block of G+−+ have a vertex in common. This implies B(G+−+) = L(G) .

Conversely, suppose B(G+−+) = L(G) . If G has a cycle, then it is easy to see that the number of vertices in B(G+−+) is less than that in L(G) , hence B(G+−+) ≠ L(G). This is a contradiction.

4 CONCLUSIONS

In this paper, we present basic properties, covering invariants, connectivity and characterization of graphs whose transformation graph G+−+ are eulerian. We have also derived some properties of the transformation graphs.

REFERENCES

[1] J. Akiyama, T. Hamada and I. Yoshimura, Miscellaneous properties of middle graphs, TRU Mathematics, 10 (1974), 41-53.
[2] S. Arumugam and S.V. Velammal, Edge domination in graphs, Taiwanese Journal of Mathematics, 2(2)(1998), 173-179.
[3] R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, Springer-Verlag, New York, Inc.(2012).
[4] B. Basavanagoud and Sujata S. Tallur, Further Results on Degree Splitting graph of a graph, Acta Ciencia Indica, Vol.XXXIII M, 4(2007), 1403-1414.
[5] B. Basavanagoud, K. Mirajakar and S.H. Malghan, On trnas formation graph Gxyz when xyz = + + − , (Proc) Graph theory and its applications, Macmillan Publishers, India Ltd., (2009), 123-130.
[6] B. Basavanagoud, K. Mirajakar and S.H. Malghan, Traversability and planarity of the trnasformation graph Gxyz when xyz = + +, (Proc) Graph theory and its applications, Macmillan Publishers, India Ltd., (2009),153-165.
[7] B. Basavanagoud and S.M. Hosamani, Mediate dominating graph of a graph, International Journal of Mathematical Combinatorics, Vol. 3 (2011), 68 – 75.
[8] B. Basavanagoud and S.M. Hosamani, Complete cototal domination in graphs, J. Sci. Res., 3(3) (2011), 547-555.
[9] B. Basavanagoud and S.M. Hosamani, The middle dominating graph of a graph, Int. J. Contep. Math. Sciences, Vol.5, No.55 (2010), 2709-2715.
[10] M. Behzad, A criterion for the planarity of a total graph, Proc. Cambridge Philos. Soc., 63(1967), 679-681.
[11] C. Berge, The theory of graphs and its applications, Methuen, London (1962).
[12] J.A. Bondy and J.S.R. Murty, Graph theory with applications, American Elsevier. Publishing Co., INC (1977).
[13] G. Chartrand and F. Harary, Planar permutation graphs, Ann. Inst. Henri poincare' Sec. B3 (1967), 433-438.
[14] G. Chartrand and L. Lesniak, Graphs and digraphs, Fourth edition, Wadsworth and Brooks/Cole, Pacific Groove, California(2005).
[15] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in Graphs, Networks, 7(1977), 247-261.
[16] E. J. Cockayne, R.M.Dawes and S. T. Hedetniemi, Total domination in graphs, Networks, 10(1980), 211-219.
[17] F. Harary, Graph theory, Addison-Wesley, Reading, Mass, (1969).
[18] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc, New York (1998).
[19] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in graphs: Advanced topics, Marcel Dekker, Inc., New York (1998).
[20] T. N. Janakiraman, S. Muthammai and M. Bhanumathi, Global domination and neighborhood numbers in boolean function graph of a graph , Mathematica Bohemica, Vol. 130(3) (2005), 231-246.
[21] V.R.Kulli and B.Janakiram, The split domination number of a graph, Graph Theory Notes of New York, New York Academy of Sciences, 32(1997), 16-19.
[22] V.R.Kulli and B.Janakiram, The maximal domination number of a graph, Graph Theory Notes of New York, New

York Academy of Sciences, 33(1997), 11-13.
[23] V.R. Kulli, B. Janakiram and R.R. Iyer, The cototal domination number of a graph, J. Discrete Mathematical Sciences and Cryptography, 2 (1999), 179 – 184.
[24] V.R. Kulli, Theory of domination in graphs, Vishwa Inter. Publ. (2010).
[25] Lei Yi and B. Wu, The transformation graph G++− , Aust. J. of Combi., Vol.44(2009), 37-42.