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ON SOME STRONGLY HARMONIC GRAPHS

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ABSTRACT

In this paper, we introduce the concept of strongly harmonic graphs and discuss some properties and results of strongly harmonic graphs. We also estimate the bounds for the maximum number of edges in a strongly harmonic graph of order n.

1 INTRODUCTION

A graph G consists of a nonempty set V = V (G) of points called vertices and another set E = E(G) whose elements are called edges where each edge is identified with an unordered pair of vertices in V. Each pair e = (u, v) in E of points of V is an edge of G and is said to join u and v. We write e = uv and say that u and v are adjacent vertices, the vertex u and the edge e are incident with each other, as are v and e. If two distinct edges e1 and e2 are incident with a common vertex then they are called adjacent edges. If an edge is associated with a pair (v, v), that is, having the same vertex v as both of its end vertices, then it is called a loop. It is also possible that more than one edge can be associated with a given pair of vertices. Such edges are referred to as parallel edges. A graph that has neither loops nor parallel edges is called a simple graph. A graph with n vertices and m edges is called an (n,m) graph.

A graph labelling is an assignment of integers to the vertices or edges or both subject to certain conditions. After it was introduced in late 1960's thousands of research articles on graph labelling and their applications have been published. Labelled graphs serve as useful models for

a broad range of applications such as coding theory, X-ray crystallography, radar, Astronomy, circuit designs, communication designs, data base management and models for constraint programming over finite domains.

2 PRELIMINARIES

We define a strongly harmonic graph as follows.

Definition. Let G be a graph of order n. The graph G is said to be a strongly harmonic graph if there is an injective mapping $f : V(G) \to \{1, 2, \ldots, n\}$ such that the induced mapping $fh : E(G) \to Q$ defined by

$$fh(e) = f(u)f(v)/ f(u) + f(v),$$

where e = uv, is injective.

For example the following graphs are strongly harmonic.

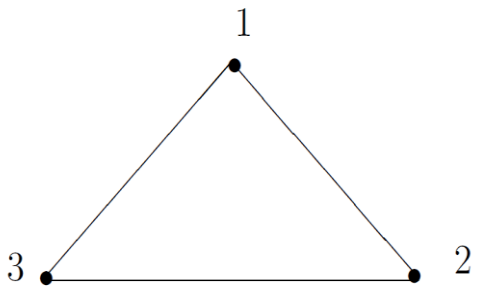

Figure1

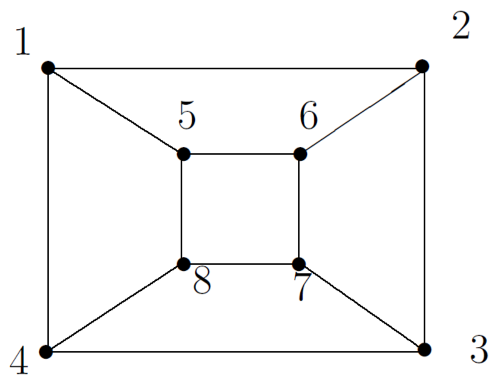

Figure2

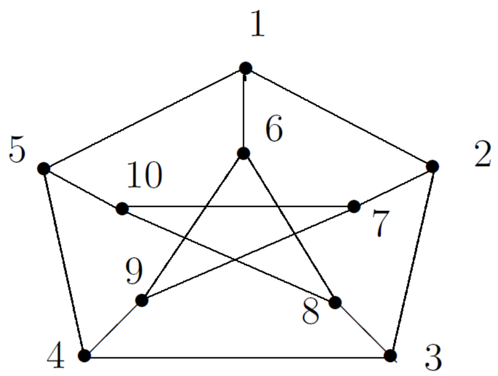

Figure 3

3 SOME CLASSES OF STRONGLY HARMONIC GRAPHS

In this Section, we show that the complete graph K_n, for $n \geq 11$, all cycles, wheels, trees, grids, ladders, triangular ladders, fans, stars, double stars, the graph $K_2 + mK_1$, cycles - cactus, triangular snakes and Mycielskian graph of the path are all strongly harmonic graphs. In Section 2.3, we found the upper and lower bounds for the maximum number of edges in a strongly harmonic graph of order n.

We now investigate which classes of graphs are strongly harmonic graphs.

Theorem 1. The complete graph Kn is strongly harmonic if and only if n ≤ 11.

Proof. For n ≤ 11, it is easy to see that Kn are strongly harmonic graphs. When n = 12, we have 4 · 3/4 + 3=12/7=24/14=12 · 2/12 + 2. Therefore K 12 is not strongly harmonic graph and hence any complete graph K n , for n ≤ 12 is not strongly harmonic.

Theorem 2. Every cycle Cn is a strongly harmonic graph.

Proof. Let C n = [v1, v2, . . . , vn, v1] be a cycle of order n. Then consider the following labelling of the graph f(v1) = 1, f(v2) = 2, . . . , f(vn) = n. Then the value of the edge vkvk+1 is fh(vkvk+1) = k(k+1)/2k+1 , for 1 ≤ k < n. The value of the edge vnv1 is fh(vnv1)= n/n+1. Since 2/3 < n/n+1 < 6/5 < · · · < n(n−1)/2n−1 , for all n ≥ 3,it follows that the values of the edges are all distinct, proving that every cycle Cn, n ≥ 3, is strongly harmonic.

Theorem 3. Every tree is a strongly harmonic graph.

Proof. Label the vertices of the tree using breadth-first search method. To show that the labeling is strongly harmonic it suffices to consider the following two cases:

Case(i): Let e1 = (a, b) and e2 = (a, c) be the edges with a common vertex as shown below.

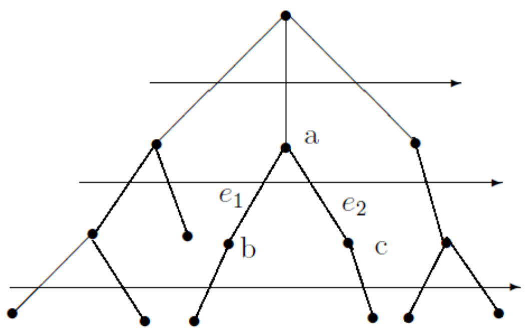

Figure4

From the breadth-first search method of labeling it follows that $a<b<c$. This implies that $ab/a+b < ac/ a+c$. Hence the values of the edges with common vertex form a strictly increasing sequence of rational numbers.

Case(ii): Let e1 = (a, c) and e2 = (b, d), where the edges e1 and e2 fall in the same level.

From the breadth-first search method of labeling it follows that $a<b<c<d$. This implies that $ac/a+c<bd/b+d$. Hence as indicated by the arrows, the values of the edges form a strictly increasing sequence of rational numbers. Thus the values of the edges are all distinct. Hence every tree is a strongly harmonic graph.

Theorem 4. Every grid is a strongly harmonic graph.

Proof. Label the vertices of the grid using breadth - first search method. To show that the labeling is strongly harmonic it suffices to consider the three cases. The first two cases are similar to the two cases considered in the proof of the Theorem 3. The last case is when e1 = (a, c) and e2 = (b, c) as shown in the Figure 5.

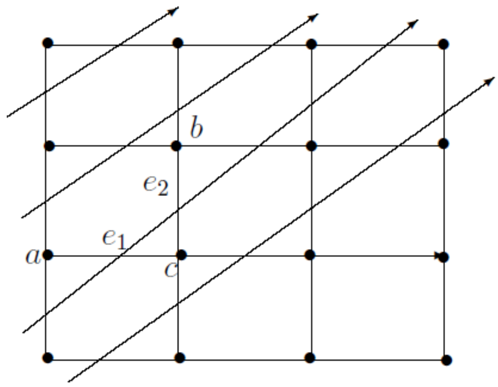

Figure 5

In this case from the breadth-first search method of labeling it follows that a<b <c which implies that ac/a+c<bc/b+c. Therefore, as indicated by the arrows, the values of the edges form a strictly increasing sequence of rational numbers. Thus, the values of the edges are all distinct proving that every grid is strongly harmonic.

Theorem 5. Every ladder Ln is a strongly harmonic graph.

Proof. Consider the ladder Ln = P2 × Pn. Let the vertices of Ln be u1, u2, . . . , un and v1, v2, . . . , vn as shown in the Figure 6.

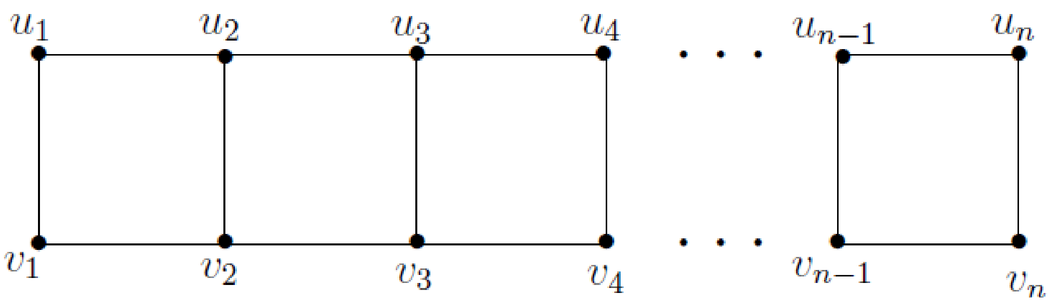

Figure6

We label the vertices of L n as follows:

f(ui) = 2i − 1, f(vi) = 2i, for 1 ≤ i ≤ n.

Then the value of the edge uiui+1 is fh(uiui+1) = (2i−1)(2i+1)/4i , for 1 ≤ i ≤ n − 1, the value of the edge vivi+1 is fh(vivi+1) = 2i(2i+2)/4i+2 , for 1 ≤ i ≤ n − 1 and the value of the edge uivi is fh(uivi) = 2i(2i−1)/4i−1 , for 1 ≤ i ≤ n. Further, we have, for 1 ≤ i ≤ n − 1,

2i(2i − 1)/4i − 1<(2i − 1)(2i + 1)/4i<2i(2i + 2)/4i + 2<(2i + 2)(2i + 1)/4i + 3.

Thus the values of the edges of the ladder are all distinct, proving that the ladder is a strongly harmonic graph.

Theorem 6. Every star K1,n is a strongly harmonic graph.

Proof. Consider the star graph K1,n. Let c denote the center of the star and v1, v2, . . . , vn be the vertices. We label the vertices of K1,n as follows:

f(c) = 1, f(vi) = i + 1, for 1 ≤ i ≤ n.

Then the value of the edge cvi is f_h(cvi)=1·(i+1)/i+2, for 1 ≤ i ≤ n and 1·(i+1)/i+2<1·(i+2)/i+3,for1 ≤ i ≤n −1. Hence values of the edges of the star are all distinct, proving that star is a strongly harmonic graph.

Theorem 7. Every double star is a strongly harmonic graph.

Proof. Consider a double star graph G obtained by joining the centers of two disjoint stars with an edge. Let c1 and c2 be the centers of the double star. Let v1,1, v1,2, . . . , v1,n and v2,1, v2,2, . . . , v2,n be the vertices of the double star. We label the vertices of double star as follows:

f(c1) = 1, f(c2) = 2, f(v1,i) = 2i + 2, f(v2,i) = 2i + 1 for 1 ≤ i ≤ n.

Then the value of the edge $c_1v_{1,i}$ is $f_h(c_1v_{1,i})$ =1·(2_i+2)/2_i+3 , for 1 ≤ i ≤ n and the value of the edge $c_2v_{2,i}$ is $f_h(c_2v_{2,i})$ =2·(2_i+1)/2_i+3 , for 1 ≤i ≤ n. Then we have, for1 ≤ i ≤n,

1· (2i + 2)/2i+3<1 · (2i+4)/2i +5;2 · (2i + 1)/2i +3<2 · (2i+3)/2i +5.

Also 1 · (2n + 2)/2n + 3<6/5. Hence the value of the edges of the double star G are all distinct, proving that the double star is a strongly harmonic graph.

Definition. For a graph G = (V,E), the Mycielskian of G is the graph µ(G) with vertex set consisting of the disjoint union V ∪ V J ∪ {w}, where V J ={vJ : v ∈ V }and edge set

E ∪ {vJ u : vu ∈ E} ∪ {vJ w : vJ ∈ V J }.

The vertex vJ is called the twin of the vertex v and the vertex w is the root of µ(G).

Theorem 8. For all n ≥ 2, the Mycielskian graph µ(Pn) of the path Pn is strongly harmonic.

Proof. Let the vertices of µ(Pn) be w, u1, u2, . . . , un and v1, v2, . . . , vn as shown in Figure 7.

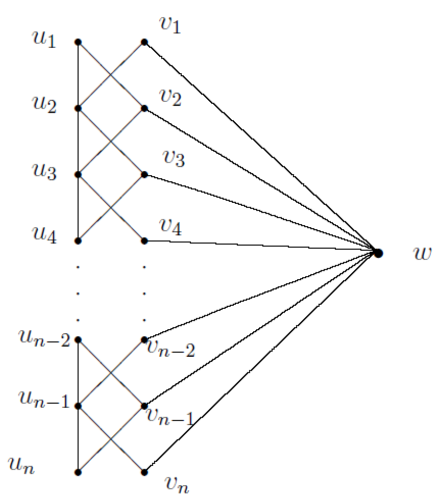

Figure 7

We label the vertices of $\mu(P_n)$ as follows:
$f(w) = 1$, $f(u_i) = 2i + 1$, $f(v_i) = 2i$, for $1 \leq i \leq n$.

Then the value of the edge u_iu_{i+1} is $f_h(u_iu_{i+1}) = (2i+1)(2i+3)/4i+4$, for $1 \leq i \leq n - 1$, the value of the edge u_iv_{i+1} is $f_h(u_iv_{i+1}) = (2i+1)(2i+2)/4i+3$, for $1 \leq i \leq n - 1$, the value of the edge wv_i is $f_h(wv_i) = 2i(2i+3)/4i+3$, for $1 \leq i \leq n - 1$ and the value of the edge wv_i is $f_h(wv_i) = 1\cdot 2i/2i+1$, for $1 \leq i \leq n$. Further, we have, for $1 \leq i \leq n - 1$,

$1 \cdot 2i/2i +1<1 \cdot (2i + 2)/2i +3$ and $1\cdot 2n/2n + 1<10/7$.

For $1\leq i \leq n-1$

$2i(2i +3)/4i +3/<(2i + 1)(2i + 2)/4i+3<(2i +1)(2i + 3)/4i+4<(2i +5)(2i+2)/4i +7$.

Hence the values of the edges of $\mu(P_n)$ form an increasing sequence: Thus the values of the edges of $\mu(P_n)$ are all distinct, proving that $\mu(P_n)$ is a strongly harmonic graph.

4 CONCLUSIONS

In this paper, we have derived some properties of harmonic graphs. We have also proved certain results related to strongly harmonic graphs. We introduced the concept of strongly harmonic graphs. We also found the upper and lower bounds for the maximum number of edges in a strongly harmonic graph of order n. We familiarized the concept of square sum labeling of graphs.

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