Risk Measures for Risk-less Investments: A Verification

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Abstract
This paper verifies that the standard definition of risk measures value-at-risk (VaR), expected shortfall (ES) and spectral risk measure (SRM) are intuitively informative even when the underlying is a risk free asset. It then follows that in case of risk less asset, coherent and non-coherent risk measures do not distinguish themselves.

Keywords: Value-at-risk, Expected shortfall, Spectral risk measure, Risk aversion function

Introduction
Risk management in financial establishment is a relatively recent development and is increasingly being recognized as a discipline where both practitioners and academics find common grounds to co-operate. Last decades saw enormous growth in this literature, particularly on the face of grinding financial crisis world-wide. The risk measures which play a key role in the growth of this literature were basically developed for financial time series risk,see Dowd(2005); however subsequently these measures were studied for other time series as well such as agricultural time series.

In this paper we discuss the traditional risk measure VaR and its coherent versions ES and SRM; and verify that the definitions of the risk measures are informative even when the underlying is a risk free asset for which coherent and non-coherent distinction diminishes. A comprehensive practical survey of these risk measures can be found in Dowd (2005), Christoffersen (2003) [2]. Cotter and Dowd (2006) [3], studied these risk measures with an application to fixing clearing house margin requirement. Cotter and Dowd (2006) consider extreme value (EV) model particularly suitable in this context. More recently Sorwar and Kevin (2010)[7],further studied these risk measures in option model framework under CEV dynamics. All these are applications when the underlying are risky assets. Nonetheless, through trivial, the justification of the definitions for risk free asset is mathematically established in this paper.

0.0.1 Value-at-Risk (VaR)
Under the consideration of static version of risk measure, VaR for a given fixed time period and coverage provides us an estimate of the magnitude of the expected potential loss. In plain words it provides us, over a specific time interval and for a given level, the worst expected loss under normal market condition. So clearly VaR has three parameters, as mentioned earlier; relatively high level of confidence $(1-\alpha)$ (typically

95% or 99%), time period of projection T, (day, month,year) and the estimate of investments loss L. If X_0 and X_T denote the values of the investment at time 0 and T, respectively, the loss function is defined as follows:

$$L = X_0 e^{rT} - X_T \qquad (1)$$

Where " r " is the constantrate of interest. Then formally VaR can be defined as:

Definitions 0.1 *VaR of any risky investments at the confidence level (*$1-\alpha$*), for* $\alpha \in (0,1)$*, is given by the smallest number "D" such that the probability that the loss "L" exceeds "D" is not greater that* α *:*

$$VaR_T^\alpha = \inf\{D|P(L > D) < \alpha\}$$
$$= \inf\{D|P(X_0 e^{rT} - X_T > D) < \alpha\}$$

So according to Christoffersen (2003) [2], VaR answers the question " what dollar loss is such that it will only be exceeded $\alpha \times 100\%$ of the time in next "T" trading days ?"

Similarly, in case of portfolio, if we denote the portfolio return as R_{PF} then we can write: $\$L = -X_0 * R_{PF}$.

Then equation (2) implies:

$$P(-X_0 * R_{PF} > VaR_T^\alpha) = \alpha \qquad (3)$$

That is:

$$P\left(R_{PF} < \frac{VaR_T^\alpha}{-X_0}\right) = \alpha$$

Thus defining VaR with respect to the coherent value of the portfolio, X_0, we write:

$\overline{VaR_T^\alpha} = \frac{VaR_T^\alpha}{X_0}$. Then from equation (3) we obtain:

$$P\left(R_{PF} < \overline{VaR_T^\alpha}\right) = \alpha \qquad (4)$$

As in [?] writing VaR relative to the current value of the portfolio makes it much easier to think about. E.g. knowing that the VaR_T^α is equal to $50000 doesn't mean much unless we know the current value of the portfolio.

However knowing that VaR_T^α is 50% of the value of the portfolio conveys much more information.

Let $\overline{VaR_T^\alpha}$ denote the 1% ($\alpha = 0.01$) VaR for the one day ahead return (T=1). If returns are normally distributed with mean zero and standard deviation $\sigma_{PF,T}$ then it can be shown, see e.g. Down(2005) [4], Christoffersen (2003) [2], that:

$$\overline{VaR_T^\alpha} = -\sigma_{PF,T} \times \Phi_\alpha^{-1}$$
$$= -\sigma_{PF,T} \times (-2.33) \qquad (5)$$

So the only information we need to obtain the VaR is tomorrow's (T=1) variance forecast. As Φ_α^{-1} is always negative for $\alpha < 0.5$, the minus sign in front of the VaR formula ensures that the VaR itself is a positive number. Thus in precise terms considering one day ahead uncertainty (T=1), the one day VaR with 99% coverage gives us a number VaR_T^α such that there is 1% chance of losing more than $[\sigma_{PF,T} \times (-2.33)] \times 100\%$ of the today's portfolio value. However this simplicity is because of the assumption of normality in return and for non normal models things are not so straightforward.

However, VaR has serious drawbacks. One of the remarkable drawbacks is that it ignores the extreme losses. According to equation (4), it only tells us that 1% of the time we will get a return below the reported VaR number $\overline{VaR_T^\alpha}$, but it says nothing about what will happen in those 1% worst cases. Thus in theoretical terms, among others two key desirable characteristics of a risk measure , namely coherence and subadditivity, are not satisfied by VaR.VaR fixes tail events corresponding to a given confidence level but leaves the tail completely unattended. Knowing an amount of possible loss to the occurrence of extreme event is important but what is more

important is to have the idea of how catastrophic the loss could be once such an event occurs. Moreover VaR assumes that the portfolio is constant in next "T" trading days which is unrealistic in many cases when "T" is larger than a day or a week. Finally it may not be clear how one should choose "T" and "α".

Despite all its limitations the tail based risk measure VaR has seen a great amount of applications in many stochastic environments where risk management makes a difference and undoubtedly becomes the industry benchmark for risk calculation. This is because it captures the important aspect of risk namely how bad things can get with a certain probability, α. Also it can be easily communicated and understood.

However we have a different focus on the discussion on VaR which further reinforces the underlying intuition.

Proposition 0.1 *VaR of a risk free asset is zero.*

Proof: We know in case of risk free asset, say bond, we have:

$$B_t = B_0 e^{rt} \qquad t \in [0, T]$$

Since sources of randomness "z" has no role to play in case of risk free asset, the loss function "L" as defined in (1), turns out to be:

$$L(z) = B_0 e^{rT} - B_T(z) = 0$$

Therefore without loss of generality we can assume that the constant random variable "L(z)" can be described by a distribution of the form:

$$P(L(z) \leq 1) = \begin{cases} 1 & if\ 1 > 0 \\ 0 & otherwise \end{cases}$$

(6)

Now applying to the definition of VaR, we obtain:

$$VaR_T^\alpha = \inf\{D | P(L > D) < \alpha\}$$
$$= \inf\{D | P(L \leq D) > 1 - \alpha\}$$
$$= \inf\left\{\{D \leq 0 | P(L \leq D) > 1 - \alpha\} \bigcup \{D > 0 | P(L \leq D) > 1 - \alpha\}\right\}$$

$$= \inf\{\{D \leq 0 | 0 > 1 - \alpha\} \cup \{D > 0 | 1 > 1 - \alpha\}\}$$
     [using (6)]
$$= \inf\{(0, \infty) \cup \Phi\}$$
$$= 0$$

The proof is complete.

0.0.2 Expected shortfall

Mathematically it can be argued that for quantile based estimate it is possible to have similar 1% VaR for two portfolios having completely different 0.1% or 0.01% VaR. See Christoffersen (2003)[2]. That is VaR estimate with 1% coverage rate completely fails to reveal the fact that the tail shapes of the distributions may be completely different corresponding to portfolios with different exposures. This translates into the great limitation ofVaR namely it concerns only with number of losses exceeding the VaR but not the magnitude of losses. However the magnitude should be of serious concern to risk manager as large VaRexceedence are much more likely to cause financial distress, such as bankruptcy, than those of small exceedence. Thus a risk measure accounting for both frequency and magnitude of large losses is very much expected. This is exactly what Expected shortfall does. Once modeled correctly the tail of the portfolio return distribution bears significant information to risk managers about the future losses. But with VaR, getting the idea of the shape of the entire tail of the return distribution is equivalent to computing it for various coverage levels, which is certainly less effective as a reporting tool. Expected shortfall bears this significance as a convenient reporting tool. It has the formal mathematical expression as:

$$ES_{t+1}^p = -\mathbb{E}_t[R_{t+1} | R_{t+1} < -VaR_{t+1}^p]$$

(7)

The negative sign in front of the VaR and expectation signifies that both VaR and expected shortfall are defined as positive numbers. Hence the underlying intuition is

that expected shortfall represents the expected value of those future returns'which are worse thanVaR.The tail losses and its distribution can be thought of as a two dimensional object which gives us the information about the range of possible losses along x-axis and probability associated with each outcome along y-axis. The measure expected shortfall aggregates these two dimensions into a single number by computing the average of the tail outcomes weighted by their probabilities. Thus when VaR gives us the loss such that 1% of the extreme losses will be worse that it, ES gives us the expected value of those extreme losses exceeding the VaR. Thus the up-shoot is that though ES is not providing complete information about the shape of the tail, the shape beyond the VaR measures, however, is now being accounted in quantifying the risk. Expected shortfall, coherent version of VaR, is developed to provide the investors the idea of how severe the loss could be, on an average, once extreme event occurs.

Corollary 0.1 *The ES of a risk less investment is zero.*

The proof is obvious from the definition of ES and the Proposition 0.1.

0.0.3 Spectral Risk Measure (SRM)

Expected shortfall assigns equal weight to the losses in excess of VaR, which doesn't reflect investors relative risk appetite. Instead one can define more general risk measure M_ϕ that are weighted averages of quantiles of the loss distribution:

$$M_\phi = \int_0^1 \phi(p) VaR(p)\, dp$$

(8)

Here ϕ is a general weighting function which, in its general forms, assigns different weights to different quantiles reflecting investors' appetite for underlying risk. The VaR and ES are the special cases of SRM. For the SRM to correspond to ES the

weighting function ϕ is required to have the following form:

$$\phi(p) = \begin{cases} 0 & p < \alpha \\ 1 & \\ \dfrac{1}{1-\alpha} & p \geq \alpha \end{cases}$$

VaR being a single quantile corresponds to the SRM when ϕ is the Dirac delta function putting all the mass to the particular event $\{p = \alpha\}$ and zero mass to all other events $p \neq \alpha$.So by its very definition VaR ignores the quantiles in the tail as it assigns zero mass to all quantiles other than VaR itself where as ES assigns equal mass of $\dfrac{1}{1-\alpha}$ to all quantiles in the tail specified by the VaR.

Roughly speaking a coherent risk measure basically ensures that higher losses are assigned the weights which are at least not less than any weight assigned to lower losses. To ensure this the weighting function ϕ is required to satisfy the following conditions:

- Non-negativity :
  $$\phi(p) \geq 0, \quad \forall p \in [0,1]$$
- Normalization : $\int_0^1 \phi(p)\, dp = 1$
- Monotonicity : for any two $p_1, p_2 \in [0,1]$ with $p_1 \geq p_2, \phi(p_1) \geq \phi(p_2)$

See e.g. Acerbi (2004)[1]. Clearly the monotonicity reflects the investors risk averse attitude. However if there is no risk, we have nothing to worry about:

Corollary 0.2*Forany risk less investment the SRM is identically zero for any choice of weighting function.*

The proof follows from the definition of SRM and the Proposition 0.1

Conclusion

We revisit the basic ideas behind the risk measure VaR and study how more

sophisticated risk measures like expected shortfall and spectral risk measures can be viewed as rich consistent tools for quantifying risk. We analyze that VaR as a simple reporting tool doesn't tell us anything about the underlying risk beyond the coverage leveland its coherent versions ES and SRM are more informative, though little complicated, in describing the underlying risk scenario with more flexibility to model such risk. This paper shows that in case of risk less asset, however, all sophisticated interpretations boil down to the ground case; i.e. the level of risk quantifications under all risk measures is zero,a verification to the universal definitions of all risk measures.

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