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The study of SUSY Heisenberg Uncertainty Products using supersymmetric quantum mechanical approach

Dr. V.K. Srivastava*

Department of Physics, K.I.P.M. - College of Engg. & Technology GIDA, Gorakhpur – 273209, U.P., India.
Corresponding author: *Dr. V.K. Srivastava, Department of Physics, K.I.P.M. - College of Engg. & Technology GIDA, Gorakhpur – 273209, U.P., India.

Abstract
Supersymmetric quantum mechanics has been developed as an elegant analytical approach to one dimensional problems. It generalizes the ladder operator approach used in the study of the harmonic oscillator. In this treatment, the factorization of a one dimensional Hamiltonian obtained using “charge operators”. For 1D harmonic oscillator, lowering and raising charge operators can be used. It not only allows the factorization of 1D Hamiltonian but also form Lie algebraic structure which generates isospectral SUSY partner Hamiltonians. In addition several different approaches have been employed to study generalized and approximate coherent states of systems other than harmonic oscillator. In this paper, algebraic treatments being applied to the extension of coherent states for shape-invariant systems and SUSY Heisenberg uncertainty products.

Keywords: SUSY, Heisenberg Uncertainty Principle and charge operators

Introduction

The eigenstates of the various partner Hamiltonians are connected by applications of the charge operators. As an analytical approach, SUSY- QM approach has been utilized to study a number of quantum mechanical problems including the Morse oscillator [6] and the radial hydrogen atom equation. It can be used in discovery of new exactly solvable potentials. The harmonic oscillator is fundamental to a wide range of physics including the electromagnetic field, spectroscopy, solid state physics, coherent state theory and SUSYQM. The broad application of the harmonic oscillator stems from lowering and raising ladder operators which can be used to factor the Hamiltonian of the system. For example, cannonical coherent states are defined as the eigenstates of the lowering operator of the harmonic oscillator and they are also minimum uncertainty states which minimize the Heisenberg uncertainty product for position and momentum. The lowering operator of the harmonic oscillator annihilates the ground states and it minimizes the HUP (Heisenberg Uncertainty Product).

Conventional harmonic oscillator coherent states correspond to those states which minimizes the position – momentum uncertainty relation. However, these harmonic oscillator coherent states are also constructed by applying shift operators labelled with points of the discrete phase space to the ground state of the harmonic oscillator, known as " fiducial state "[7]. But, Klauder and Skagerstam choose to define coherent states in broadest sense. Similarly, the charge operator in SUSY-QM annihilates the ground state of the corresponding system. Let us construct system-specific coherent states for any bound quantum system by similarity between treatment of harmonic oscillator and supersymmetric quantum mechanics. Since the charge operators annihilates the ground state, the superpotential that arises in SUSY QM can be regarded as a SUSY displacement operator or a generalized displacement variable.

SUSY Heisenberg Uncertainty Products

The charge operator annihilates the corresponding ground state

$$Q\psi_0 = (\bar{W} + i\bar{p}_x)\psi_0 = 0 \qquad \text{…….(1)}$$

For the harmonic oscillator, the charge operators correspond to the raising and lowering operators for the harmonic oscillator with $W(x) = x$. By the similarity, the super potential $\bar{W}$ can be regarded as a "SUSY-displacement" operator although such a displacement would, in general, not be generated by the standard momentum operator $\bar{p}_x$. In fact, $\bar{W}$ and $\bar{p}_x$ are not cannonically conjugate variables.

The ground state of the harmonic oscillator is a minimum uncertainty state, which miminizes the Heisenberg uncertainty product $\Delta\bar{x}\,\Delta\bar{p}_x$. Similarly, it is expected that the ground state for a bound quantum system minimizes the SUSY Heisenberg uncertainty product $\Delta\bar{W}\Delta\bar{p}_x$.

For an arbitrary normalized wave function, we consider the square of the SUSY- displacement-standard momentum uncertainty product

$$(\Delta\bar{W}\Delta\bar{p}_x)^2 = \langle\psi\left|\bar{W}_1^{\,2}\right|\psi\rangle\,\langle\psi|\bar{p}_{1x}^2|\psi\rangle, \qquad \text{….(2)}$$

where $\bar{W}_1 = \bar{W} - W_o \; and \; \bar{p}_{1x} = \bar{p}_x - p_o$.

The quantities $W_o = \langle W\rangle$ and $p_o = \langle\bar{p}_x\rangle$ correspond to the averaged SUSY-displacement and momentum values, respectively. In order to obtain a lower bound on the uncertainty product in equation (2), we apply the Cauchy-Schwarz inequality

$$\langle\psi|\bar{W}_1^{\,2}|\psi\rangle\,\langle\psi|\bar{p}_{1x}^2|\psi\rangle \geq |\langle\psi|\bar{W}_1\bar{p}_{1x}|\psi\rangle|^2 \qquad \text{…….(3)}$$

The equality is satisfied when the two vectors $\bar{W}_1|\psi\rangle \; and \; \bar{p}_{1x}|\psi\rangle$ are collinear. From this condition, we obtain $\bar{W}_1|\psi\rangle = \lambda\bar{p}_{1x}|\psi\rangle$.

On rearranging we get

$$(\bar{W} - \lambda\bar{p}_x)|\psi\rangle = (W_o - \lambda p_o)|\psi\rangle \qquad \text{...…..(4)}$$

As a special case for $\lambda = -i$, this equation becomes

$$(\bar{W} + i\bar{p}_x)|\psi\rangle = (W_o + ip_o)|\psi\rangle \qquad \text{...…..(5)}$$

It follows from equation (1) that $(W_o + ip_o) = \langle\psi_0|\bar{W} + i\bar{p}_x|\psi_0\rangle = 0$ for the ground state of the system. Thus, equation (1) implies that the ground state satisfies the condition in equation (5). Therefore, the ground state of a bound quantum system minimizes the SUSY- displacement-standard momentum uncertainty product $\Delta\bar{W}\Delta\bar{p}_x$.

Let us present some properties of the averaged SUSY-displacement and standard momentum values for the ground state. The averaged SUSY-displacement for the ground state is evaluated as

$$W_o = \langle\psi_0|W|\psi_0\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)W(x)\psi_0(x)dx = \int_{-\infty}^{\infty} \psi_0^*(x)\frac{d\psi_0(x)}{dx}dx, \qquad ...(6)$$

The averaged momentum for the ground state is given by

$$p_o = \langle\psi_0|\bar{p}_x|\psi_0\rangle = -i\int_{-\infty}^{\infty} \psi_0^*(x)\frac{d\psi_0(x)}{dx}dx, \qquad ...(7)$$

Again from equations (6) and (7), we have $W_o + ip_0 = 0$ for the ground state of the system, as shown in equation (1). Furthermore, when the ground state wave function is purely real, it follows from integration by parts that the integral in equations (6) and (7) is equal to zero. Thus, the averaged SUSY-displacement and momentum values for the real-valued ground state wave function are equal to zero i.e. $W_o = p_0 = 0$.

The ground state of a quantum system is the minimizer of the SUSY Heisenberg uncertainty product. I can derive the minimum value for the SUSY Heisenberg uncertainty product in equation (3). For the real-valued ground state wave function, $\bar{W}_1 = \bar{W} - W_o = \bar{W}$ and $\bar{p}_{1x} = \bar{p}_x - p_0 = \bar{p}_x$. Now the right side of the uncertainty product given by equation (3) becomes

$$\langle\psi_0|\bar{W}\bar{p}_x|\psi_0\rangle = i\langle\psi_0|\bar{W}^2|\psi_0\rangle, \qquad ....(8)$$

where $\bar{p}_x|\psi_0\rangle = i\bar{W}|\psi_0\rangle$ by using equation (1). Thus, the right side of the uncertainty product in equation (3) is given by

$$|\langle\psi_0|\bar{W}\bar{p}_x|\psi_0\rangle^2 = \langle\bar{W}^2\rangle^2 \qquad ....(9)$$

Similarly, the left side of the uncertainty product in equation (3) is

$$\langle\psi_0|\bar{W}^2|\psi_0\rangle\langle\psi_0|\bar{p}_x^2|\psi_0\rangle = \langle\bar{W}^2\rangle\langle\bar{W}^2\rangle \qquad ....(10)$$

Therefore, the equality in equation (3) holds for the ground state, and the SUSY Heisenberg uncertainty product is equal to $\Delta\bar{W}\Delta\bar{p}_x = \langle\bar{W}^2\rangle$.

The expectation value of $\bar{W}^2$ for the ground state is evaluated by

$$\langle\bar{W}^2\rangle = \int_{-\infty}^{\infty} \psi_0(x)W^2(x)\psi_0(x)dx = -\int_{-\infty}^{\infty} \psi_0(x)W(x)\frac{d\psi_0(x)}{dx}dx, \qquad ...(11)$$

Using integration by parts , we get

$$\int_{-\infty}^{\infty} \psi_0(x) W(x) \frac{d\psi_0(x)}{dx} dx = -\frac{1}{2} \int_{-\infty}^{\infty} \psi_0(x) \frac{dW(x)}{dx} \psi_0(x) dx, \quad \dots(12)$$

Thus, the expectation value of $\bar{W}^2$ for the ground state is equal to one half of the expectation value for the derivative of the superpotential

$$\langle \bar{W}^2 \rangle = \frac{1}{2} \langle \frac{d\bar{W}}{dx} \rangle \quad \dots.(13)$$

Moreover, the commutation relation of the SUSY-displacement and the momentum operator is

$$[\bar{W}, \bar{p}_x] = i \frac{d\bar{W}}{dx} \quad \dots.(14)$$

Therefore, the SUSY Heisenberg uncertainty product for the ground state becomes

$$\Delta\bar{W}\Delta\bar{p}_x = \langle \bar{W}^2 \rangle = \frac{1}{2} \langle \frac{d\bar{W}}{dx} \rangle = \frac{1}{2i} \langle [\bar{W}, \bar{p}_x] \rangle \quad \dots(15)$$

For the harmonic oscillator, $W(x) = x \text{ and } \frac{dw}{dx} = 1.$ We recover the conventional HUP for the ground state i.e. $\Delta x \Delta p_x = \frac{1}{2}.$ As a special case, a similar derivation has been employed to determine exact minimum uncertainty coherent states for the Morse oscillator [3].

Results and Discussion

The application of SUSY-QM to non relativistic quantum systems generalizes the powerful ladder operator approach used in the treatment of the harmonic oscillator. The lowering operator of the harmonic oscillator annihilates the ground state, while the charge operator annihilates the ground state of corresponding quantum systems. The similarity between the lowering operator of harmonic oscillator and SUSY charge operator implies that the superpotential can be regarded as a system specific generalized displacement variable. Analogous to the ground state of the harmonic oscillator which minimizes the HUP, the ground state of any bound quantum system was identified as minimizer of SUSY HUP. It was observed that such dynamically adopted coherent states yields significantly more accurate exited state energies and wave functions than were obtained with the same number of the conventional coherent states and from the standard harmonic oscillator basis.

Conclusion

The ladder operator approach of the harmonic oscillator and SUSY-QM formulation share strong similarity. This observation suggests that connection of the SUSY-QM with Heisenberg minimum uncertainty (μ^-) wavelets should be explored. The SUSY-displacement with the SUSY HUP can lead to the construction of the SUSY minimum uncertainty wavelets and the SUSY distributed approximating functions. These new functions and their potential applications in mathematics and physics are in progress.

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